PROBLEMS 1

1. A harmonic oscillator (e.g. one mode of a quantum field) has the hamiltonian

$$H = p^2/2 + \omega^2 q^2/2.$$

Consider the state

$$|\lambda\rangle = \exp(-|\lambda|^2/2) \sum_n \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle,$$

$a^\dagger$ is the creation operator appropriate to H and λ is a complex number. This state is known as a *coherent state*. Show that
(i) $|\lambda\rangle$ is normalised.
(ii) $|\lambda\rangle$ is an eigenstate of the annihilation operator a with eigenvalue λ.
(iii) The average number of quanta in this state is

$$\overline{N} = \langle\lambda|N|\lambda\rangle = |\lambda|^2$$

where N is the number operator for quanta of this oscillator.
(iv) The rms fluctuation in the number of quanta in this state is given by

$$(\Delta N)^2 = \langle\lambda|N^2|\lambda\rangle - \overline{N}^2 = |\lambda|^2.$$

(v) The expectation value of q at any time $t = 0$ is

$$\langle\lambda|q|\lambda\rangle = \frac{\lambda + \lambda^*}{\sqrt{2\omega}}$$

and that of p is

$$\langle\lambda|p|\lambda\rangle = -i\sqrt{\frac{\omega}{2}}(\lambda - \lambda^*).$$

By using the Heisenberg equations of motion, or otherwise, find the expectation values of p and q as functions of time. Comment on the significance of these results, in particular for large $|\lambda|$.

2. (a) Use the canonical commutation relations for a real scalar field ϕ and its conjugate momentum π,

$$[\pi(t,\mathbf{x}),\phi(t,\mathbf{y})] = -i\delta(\mathbf{x}-\mathbf{y}),$$

to show that the momentum operator

$$\mathbf{P} = -\int d^3\mathbf{x}\,\pi(x)\nabla\phi(x)$$

satisfies

$$[\mathbf{P},\phi(x)] = i\nabla\phi(x), \qquad [\mathbf{P},\pi(x)] = i\nabla\pi(x).$$

Hence show that any operator which can be constructed as a simultaneous power series in ϕ and π, i.e.

$$F(x) = \sum_{n,m\geq 0} c_{nm}\phi(x)^n\pi(x)^m,$$

satisfies

$$[\mathbf{P},F(x)] = i\nabla F(x).$$

(b) Under a spatial translation $\mathbf{x}\to\mathbf{x}' = \mathbf{x}+\mathbf{c}$, a scalar field transforms as

$$\phi(x)\to\phi'(x) = \phi(t,\mathbf{x}-\mathbf{c}).$$

Show that this can be represented by thev unitary transformation

$$\phi(x)\to\phi'(x) = U\phi(x)U^\dagger$$

where

$$U = \exp(i\mathbf{c}\cdot\mathbf{P}).$$