Notation: In these examples sans-serif type will be used to denote 4-vectors, while **bold** roman type will used for 3-vectors. For example the space-time location of an event is represented by $\mathsf{x} = (\mathbf{x},\ ict)$, where $\mathbf{x}$ denotes its position in space and t the time at which it occurs. Also the magnitude of a 3-vector is denoted by the same symbol in italic type, i.e. $|\mathbf{x}| = x$.

1. [Revision of the relativistic forms for the energy of a particle in terms of its momentum or velocity.]

(a) Write down the relativistic expression for the total energy $E(p)$ of a free particle of mass m moving with momentum p. What does this relation look like for a massless particle such as a photon?

(b) The relativistic kinetic energy of a particle $T(p)$ is defined as the difference between its total energy $E(p)$ and its rest energy mc^2:

$$T(p) = E(p) - mc^2.$$

Show that for small enough p, the kinetic energy is given approximately by its non-relativistic expression

$$T(p) \simeq \frac{p^2}{2m}.$$

What does "small enough" mean in this context?
[Hint: use a Taylor's series to expand $E(p)$ around $p = 0$.]

(c) Write down the relativistic expression for the total energy of a free particle of mass m moving with speed v. Hence show that the speed of a particle of mass m and energy E is given by

$$v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2}.$$

2. Sketch the curve consisting of all events in the xt-plane with constant invariant distance s from the origin, where

$$s^2 = x^2 - c^2t^2 \qquad \text{and} \qquad s^2 > 0.$$

3. Show that the square of the magnitude of the 4-velocity v of a particle is $-c^2$ for all values of the 3-velocity.

4. The space-time coordinates of a particle according to inertial observers O and O' are related by the equations

$$\begin{aligned} x_1' &= x_1, \\ x_2' &= x_2, \\ x_3' &= \frac{x_3 - Vt}{\sqrt{1 - V^2/c^2}}, \\ t' &= \frac{t - Vx_3/c^2}{\sqrt{1 - V^2/c^2}}. \end{aligned}$$

If the particle has momentum $\mathbf{p} = (p_1, p_2, p_3)$ and energy E according to observer O, what are its momentum and energy according to observer O' ?

If the particle is a photon, find the relation between the frequencies of the photon as perceived by observers O and O' when it is moving (a) in the $\mathbf{e}_3$ direction and (b) in the $\mathbf{e}_1$ direction.

5. A rocket of initial mass M_0 starts from rest and propels itself forwards in the $\mathbf{e}_1$ direction by emitting photons backwards. The final mass of the rocket after its engine has finished firing is M_f. All of the quantities you are asked to find are defined in the frame where the rocket started at rest.

(a) Find the total energy of the exhaust.
[Hint: Since all the exhaust is travelling in the same direction at the speed of light, you can treat it as a single "photon".]

(b) Hence write down the total 4-momentum of the rocket's exhaust.

(c) Find the final 4-momentum of the rocket.

(d) Use your result for the final energy of the rocket to show that its final speed is

$$v = \frac{1 - y^2}{1 + y^2} c,$$

where $y = M_f/M_0$ is the ratio of the rocket's final and initial masses.

6. An electron and a positron can collide and produce a proton and an antiproton,

$$e^- + e^+ \to p + \bar{p}.$$

Find the minimum kinetic energy of the positron needed for this reaction in a frame of reference in which the total momentum of the particles is zero.

Find the minimum kinetic energy of the positron needed for this reaction in a frame of reference in which the positron collides with a stationary electron.

[The masses of the electron and the positron are identical, both being 0.51 MeV/c^2. The masses of the proton and the antiproton are also identical, and are 938.3 MeV/c^2.]

7. A photon with energy above 1.02 MeV has an energy greater than the rest energy of an electron-positron pair. Nevertheless the process

$$\gamma \to e^- + e^+$$

cannot occur in the absence of other matter or radiation. Why not?

In the presence of a stationary electron, a high energy photon can produce an electron-positron pair via the reaction

$$\gamma + e^- \to e^- + e^- + e^+.$$

Find the minimum photon energy for this reaction to occur.

8. A neutral pion π^0 moving with speed $v = 0.98c$ decays in flight into two photons,

$$\pi^0 \to \gamma + \gamma.$$

If the two photons emerge on each side of the pion's direction with equal angles θ, find the energy of the photons and θ.

[The mass of the π^0 is 135 MeV/c^2.]

9. This question is about the so-called transverse Doppler effect. Consider a frame S in which a transverse wave

$$y(x,t) = \sin(kx - \omega t)$$

propagates. As seen by an observer at rest in S, this wave is propagating along the $+x$ direction with wavelength $2\pi/\lambda$ and angular frequency ω. The speed of propagation is $u = \omega/k$.

Now consider a frame S' which is moving at speed V in the $+y$ direction. The origins of S and S' coincide at $t = t' = 0$. Show that an observer in S' sees the following transverse wave:

$$y' = -Vt' + \frac{1}{\gamma}\sin(\mathbf{k}' \cdot \mathbf{x}' - \omega' t').$$

Deduce $\mathbf{k}'$ and ω', and hence show that $\mathsf{k} = (\mathbf{k}, i\omega/c)$ transforms as a 4-vector.

What is the speed of propagation in S'? Show that it reduces to the correct values in the limits $u \to c$ and $u \ll c$.

Answers

6. Minimum $T_{e^+} = 937.8$ MeV in c.m. frame

Minimum $T_{e^+} = 3.45$ TeV in frame with e^- at rest

7. Minimum $E_\gamma = 2.04$ MeV

8. $E_\gamma = 339$ MeV, $\quad \theta = 0.2$ rad