The equation of state of a relativistic, degenerate Fermi gas

We will compute the $T = 0$ approximation for the pressure P using the correct relativistic formula:

$$E^2 = c^2 p^2 + m^2 c^4 \ . \tag{1}$$

At $T = 0$ we can deduce P using

$$P = -\left.\frac{\partial U}{\partial V}\right)_N \ ,$$

which is exact at $T = 0$. So it is just a matter of deducing the internal energy of the Fermi gas:

$$U = \int \frac{d^3k}{(\pi/L)^3} \, \frac{2E}{\exp(\beta E) + 1} \ ,$$

where $p = \hbar k$ and $\beta^{-1} = k_B T$. The factor $(\pi/L)^3$ is the volume of a single particle state in k-space and the factor 2 is to account for the spin in the case of an electron gas.

At $T = 0$ the Fermi-Dirac distribution becomes a Heaviside Theta function, so that

$$U = \int_0^{k_F} \frac{L^3}{\pi^3} \frac{4\pi k^2}{8} \, 2E \, dk \ .$$

The Fermi wavenumber can itself be established in terms of the number of particles, N:

$$N = \int_0^{k_F} \frac{L^3}{\pi^3} \frac{4\pi k^2}{8} \, 2 \, dk = \frac{V}{\pi^2} \frac{k_F^3}{3} \ ,$$

with $V = L^3$.

Using Eq. (1), we have

$$U = \int_0^{k_F} \frac{V\hbar c}{\pi^2} \, k^2 \sqrt{k^2 + m^2c^2/\hbar^2} \, dk \ .$$

Let $y = \hbar k/(mc)$ and $x = \hbar k_F/(mc)$ then we can write

$$U = V \, \frac{mc^2}{\pi^2} \left(\frac{mc}{\hbar}\right)^3 \int_0^x y^2 \sqrt{y^2 + 1} \, dy \ .$$

The integral gives

$$\int_0^x y^2 \sqrt{y^2 + 1} \, dy = \frac{1}{8}((x + 2x^3)\sqrt{1 + x^2} - \sinh^{-1} x) \ ,$$

Noting that $\sinh^{-1} x = \ln(x + \sqrt{1+x^2})$ we can then write our final expression for U:

$$U = V \; \frac{mc^2}{8\pi^2} \left(\frac{mc}{\hbar}\right)^3 [(x + 2x^3)\sqrt{1+x^2} - \ln(x + \sqrt{1+x^2})] \; .$$

Now we need to take the derivative with respect to V at fixed N. We will do this by differentiating the integral directly, i.e.

$$P = -\frac{U}{V} + \left.\frac{\partial x}{\partial V}\right)_N V \; \frac{mc^2}{\pi^2} \left(\frac{mc}{\hbar}\right)^3 x^2\sqrt{x^2+1} \; .$$

Putting $\partial x/\partial V)_N = -x/(3V)$ gives

$$\begin{aligned} P &= -\frac{mc^2}{8\pi^2}\left(\frac{mc}{\hbar}\right)^3 \left[(x+2x^3)\sqrt{1+x^2} - \ln(x+\sqrt{1+x^2})\right] - \frac{1}{3}\frac{mc^2}{\pi^2}\left(\frac{mc}{\hbar}\right)^3 x^3\sqrt{x^2+1} \\ &= -\frac{mc^2}{8\pi^2}\left(\frac{mc}{\hbar}\right)^3 \left[(x+2x^3)\sqrt{1+x^2} - \ln(x+\sqrt{1+x^2}) - \frac{8}{3}x^3\sqrt{x^2+1}\right] \\ &= \frac{mc^2}{8\pi^2}\left(\frac{mc}{\hbar}\right)^3 \left[x\left(\frac{2}{3}x^2 - 1\right)\sqrt{1+x^2} + \ln(x+\sqrt{1+x^2})\right] \; . \end{aligned}$$

We can write this so that we can see explicitly the deviation from the ultra-relativistic limit, i.e. we write

$$P = Cn^{4/3} I(x) \; ,$$

where

$$C = \frac{\pi\hbar c}{2}\left(\frac{3}{8\pi}\right)^{1/3}$$

and (using $n = k_F^3/(3\pi^2) = (mcx/\hbar)^3/(3\pi^2)$)

$$I(x) = \frac{3}{2x^4}\left[x\left(\frac{2}{3}x^2 - 1\right)\sqrt{1+x^2} + \ln(x+\sqrt{1+x^2})\right] \; .$$

As illustrated in the plot on the following page, the function $I(x) \to 1$ as $x \to \infty$, which is the ultra-relativistic limit. And $I(x) \to 4x/5$ as $x \to 0$, which gives the non-relativistic result, i.e. putting $x = (3\pi^2 n)^{1/3}(\hbar/mc)$ gives

$$P = C_{NR}\, n^{5/3}$$

with

$$C_{NR} = \frac{4}{5}C\;(3\pi^2)^{1/3}\frac{\hbar}{mc} = \frac{\pi\hbar^2}{5m}(9\pi)^{1/3} \; .$$

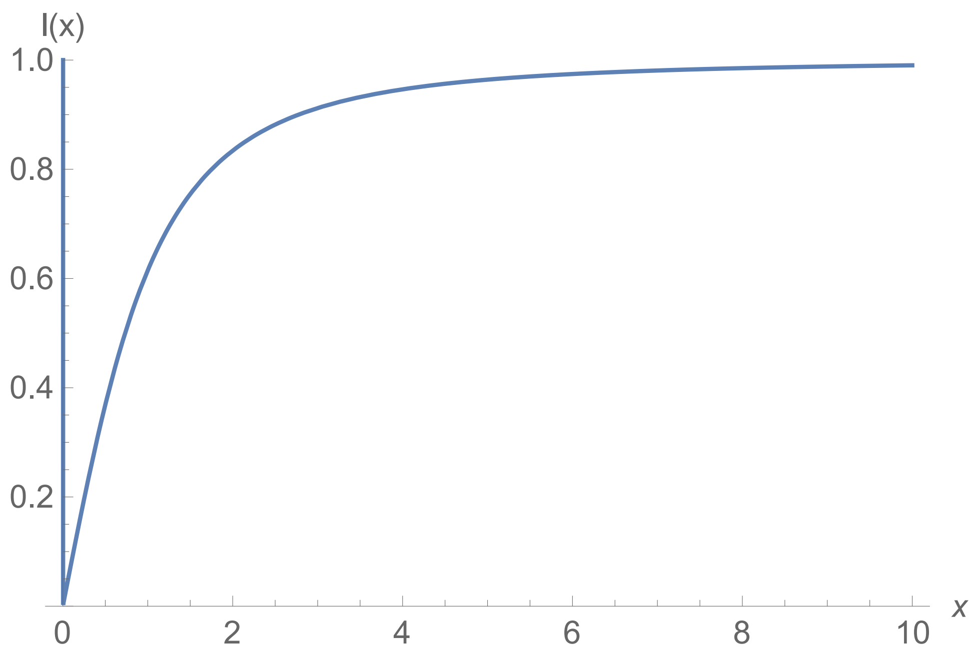

Figure 1: Plot of the function $I(x)$ against $x \propto k_F \propto n^{1/3}$.