ANGULAR POWER SPECTRUM

Starting from the temperature field $T(\theta, \phi)$ we can go ahead and compute the two-point correlation function

$$\langle T(\theta,\phi)T(\theta',\phi')\rangle = \sum_{l,m,l',m'} \langle a_{lm}a^*_{l'm'}\rangle Y_{lm}(\theta,\phi)Y^*_{l'm'}(\theta',\phi') \ , \tag{1}$$

where the angle brackets denote an average over an ensemble of possible Universes. This is obviously a theoretical ensemble since we inhabit a particular realisation but it is well defined. We shall assume that we can compute this average by averaging a single realisation over all directions. This seems ok since rotating a particular map through arbitrary angles generates an infinite subset of all possible maps with the same mean as the full ensemble. In which case we can write

$$\langle a_{lm}a^*_{l'm'}\rangle = \frac{1}{8\pi^2}\sum_{\bar{m},\bar{m}'}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta \, d\beta \int_0^{2\pi} d\gamma \ (D^l_{m\bar{m}}a_{l\bar{m}})(D^{l'}_{m'\bar{m}'}a_{l'\bar{m}'})^* \ , \tag{2}$$

where $D^l_{mm'} = D^l_{mm'}(\alpha,\beta,\gamma)$ are Wigner-D matrices and (α,β,γ) are the Euler angles for a general rotation, i.e. we use the fact that under a general rotation the a_{lm} transform as $a_{lm} \to \sum_{\bar{m}} D^l_{m\bar{m}}a_{l\bar{m}}$.

Now we can use a Schur orthogonality relation:

$$\frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta \, d\beta \int_0^{2\pi} d\gamma \ D^l_{m\bar{m}} \, D^{l'*}_{m'\bar{m}'} = \frac{1}{2l+1}\delta_{ll'}\delta_{mm'}\delta_{\bar{m},\bar{m}'} \tag{3}$$

and write

$$\langle a_{lm}a^*_{l'm'}\rangle = \frac{1}{2l+1}\delta_{ll'}\delta_{mm'}\sum_{\bar{m}}|a_{l\bar{m}}|^2 = C_l \ \delta_{ll'}\delta_{mm'} \ . \tag{4}$$

This equation is telling us that the a_{lm} are uncorrelated for different l and m and it is a result of rotational invariance, i.e. there is no assumption of gaussianity. We can now write Eq. (1) as

$$\langle T(\theta,\phi)T(\theta',\phi')\rangle = \sum_{l,m} C_l \ Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \ , \tag{5}$$

but the addition theorem for spherical harmonics says that

$$\sum_{l,m} Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') = \frac{2l+1}{4\pi}P_l(\cos\bar{\theta}) \tag{6}$$

where $\bar{\theta}$ is the angle between the two vectors defined by (θ,ϕ) and (θ',ϕ'). Thus Eq. (5) becomes

$$\langle T(\theta,\phi)T(\theta',\phi')\rangle = \frac{1}{4\pi}\sum_l (2l+1) \ C_l \ P_l(\cos\bar{\theta}) \ . \tag{7}$$

We haven't proven it but "gaussianity" implies that the set of C_l completely define the temperature field, i.e. higher correlation functions can then always be expressed as products of the two-point function.