EXAMPLES SHEET 1 : FRW UNIVERSE

1. Dynamical equations

Show that the one can derive the Raychauduri equation from

$$\dot{\rho} + 3H(\rho + P) = 0 \qquad \text{and} \qquad H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,.$$

2. Scale factor during radiation and matter domination

Compute an expression for the scale factor in terms of conformal time in a universe containing only matter and radiation with densities relative to critical given by $\Omega_{\rm m}$ and $\Omega_{\rm r}$ respectively.

3. Natural units

Convert the following to natural units:

$$\text{(a) } 10^{15}M_\odot \quad \text{(b) } 500\mu\text{Jy} \quad \text{(c) } 1000\text{km}\,\text{sec}^{-1}\,.$$

Convert the Planck density $\approx 1.6 \times 10^{57}\text{GeV}^4$ into SI units.

If an object has a mass per unit length of $(10^{16}\text{GeV})^2$, what is this in SI units?

(NB $1\text{Jy} = 10^{-26}\text{W}\,\text{m}^{-2}\text{Hz}^{-1}$)

4. FRW timescale

Write down the Friedmann equation for a flat, homogeneous and isotropic universe containing a component with energy density ρ. What conservation equation does ρ satisfy? If the energy component is pressureless matter, how does ρ depend on the scale factor $a(t)$? If the scale factor is unity at the present day, show that

$$H_0 t_0 = \frac{2}{3}$$

where H_0 is the Hubble constant and t_0 is the age of the universe. Note that the critical density at the present day is given by $\rho_{\rm crit} = 3H_0^2/8\pi G$ where G is the gravitational constant.

5. Luminosity distance

Consider a flat universe which comprises matter and a cosmological constant with densities relative to the critical density at the present day $\Omega_{\rm m}$ and Ω_Λ respectively. The luminosity distance of a source of radiation at redshift $1 + z = 1/a$ is given by

$$d_{\rm L} = (1+z)\int_0^z \frac{dz'}{H(z')}\,,$$

where $H(z)$ is the Hubble parameter at redshift z. Show that

$$H(z) = H_0 \left[1 + \frac{3}{2}\Omega_{\mathrm{m}} z + \mathcal{O}(z^2) \right] ,$$

and hence deduce the coefficient of z^2 in a power law expansion of d_{L}. Hence, deduce the value of q_0.

6. Horizon scale at Matter-Radiation Equality

Using the solutions for the scale factor of a flat universe during the matter-radiation transition, show that the conformal time at matter-radiation equality is given by $\eta_{\mathrm{eq}} = 2(\sqrt{2}-1)H_0^{-1}\Omega_{\mathrm{m}}^{-\frac{1}{2}} a_{\mathrm{eq}}^{\frac{1}{2}}$. Hence, show that the horizon at matter-radiation equality, when evolved as a comoving scale, is today equal to $16\,(\Omega_{\mathrm{m}}h^2)^{-1}$ Mpc.

7. Multipole coefficients

The observed temperature anisotropies are decomposed in terms of spherical harmonics

$$\frac{\Delta T}{T} = \sum a_{lm} Y_{lm}(\theta, \phi) .$$

Use the orthogonality of spherical harmonics to show that

$$a_{lm} = \int d\Omega \; Y_{lm}^*(\theta, \phi) \frac{\Delta T}{T}(\theta, \phi) .$$

If the anisotropies are azimuthally symmetric show that

$$a_{l0} = \pi \sqrt{\frac{2l+1}{\pi}} \int_{-1}^{1} d(\cos\theta)\, P_l(\cos\theta) \frac{\Delta T}{T}(\cos\theta)$$

and all the other multipole coefficients are zero.

8. Correlation function

For the purposes of this question use Δ to denote the density contrast $\delta\rho/\rho$. If

$$\Delta(\mathbf{x}) = \sum_i \delta\left(\mathbf{x} - \mathbf{x}_i\right) , \tag{1}$$

show that the correlation function is given by

$$\xi(\mathbf{r}) = \frac{1}{V} \sum_{i,j} \delta\left(\mathbf{r} - \mathbf{x}_i + \mathbf{x}_j\right) . \tag{2}$$

Hence, deduce an expression for $|\Delta_k|^2$.

9. Critical density in radiation and matter eras

Compute the Hubble parameter as a function of time in the radiation and matter dominated eras. Use this to deduce that the critical density is given by

$$\rho_{\rm crit} \approx \frac{3}{32\pi G t^2} \,,$$

in the radiation era and

$$\rho_{\rm crit} \approx \frac{1}{6\pi G t^2} \,.$$

in the matter era.

10. Time of the electroweak phase transition

Show that, in the radiation era, the time, t, and the temperature, T, are related by

$$t \approx 0.3 g^{-1/2} \frac{m_{\rm pl}}{T^2} \,,$$

and use this to deduce the time at which the electroweak phase transition takes place.