EXAMPLES SHEET 2 : STRUCTURE FORMATION

1. Correlation function

For the purposes of this question use Δ to denote the density contrast $\delta\rho/\rho$. If

$$\Delta(\mathbf{x}) = \sum_i \delta\left(\mathbf{x} - \mathbf{x}_i\right) \,, \tag{1}$$

show that the correlation function, $\xi(\mathbf{r}) \equiv \langle \Delta(\mathbf{x}+\mathbf{r})\Delta^*(\mathbf{x})\rangle$, is given by

$$\xi(\mathbf{r}) = \frac{1}{V}\sum_{i,j} \delta\left(\mathbf{r} - \mathbf{x}_i + \mathbf{x}_j\right) \,. \tag{2}$$

Hence, deduce an expression for $|\Delta_k|^2$.

2. Decaying modes in the Mezaros equation

If $y = a/a_{\rm eq}$ then the matter density contrast $\delta_{\rm m}$ satisfies

$$\frac{d^2\delta_{\rm m}}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_{\rm m}}{dy} - \frac{3}{2y(1+y)}\delta_{\rm m} = 0\,, \tag{3}$$

in a universe containing matter and radiation. One solution is $\delta_{\rm m} = 2/3 + y$. By setting $\delta_{\rm m} = (2/3+y)v(y)$, or otherwise, derive another solution to the differential equation and show that it has the correct asymptotic behaviour for $y \ll 1$ and $y \gg 1$.

3. Scale invariance

In order to compare the distribution of galaxies at large scales to the linear theory of density perturbations it is useful to smooth the galaxy distribution by integrating over some finite volume, i.e. for some distribution $g(\mathbf{x})$ the corresponding smoothed distribution is

$$g(R,\mathbf{x}) = \frac{1}{V}\int W(|\mathbf{x}' - \mathbf{x}|/R)g(\mathbf{x}')\; d^3\mathbf{x}' \,, \tag{4}$$

where $W(r/R)$ is called the window function and, if necessary, you may assume it is a top-hat function, i.e. $W(y) = 1$ for $y < 1$ and $W(y) = 0$ otherwise (in which case $V = \frac{4}{3}\pi R^3$).

Show that the variance (i.e. mean squared density contrast) smoothed on a scale R, σ_R^2, can be written in terms of the Fourier transform of the window function, $\tilde{W}(kR)$, such that

$$\sigma_R^2 = \int \frac{d^3k}{(2\pi)^3}\, P(k,\eta)\, |\tilde{W}(kR)|^2 \tag{5}$$

for a distribution with power spectrum $P(k,\eta)$.

If the initial power spectrum is $P_{\rm i} \propto k^n$ compute σ_R at horizon crossing (i.e. $R = \eta$) and show that it is independent of time if $n = 1$.

4. Power spectrum normalization

σ_8, the variance in the density fluctuations inside spheres of size 8 h^{-1} Mpc, is observed to be 0.8. (8 h^{-1} Mpc is smaller than the horizon size at matter-radiation equality, see problem sheet 1, question 6.) We also know that (assuming scale invariance) the CDM power spectrum is $P(k) = Bk/k_{\rm eq}^4$ for $k \ll k_{\rm eq}$ and $P(k) = Ak^{-3}$ for $k \gg k_{\rm eq}$, where A and B are dimensionless. Use this information to derive a formula for A that involves σ_8 and a dimensionless integral involving the window function. How does σ_R behave when $k_{\rm eq}R \gg 1$? What happens if the dark matter is hot?

5. Sound speed in matter and radiation universe

Consider a matter and radiation dominated universe. Derive an expression for the sound speed:

$$c_{\rm s}^2 = \frac{dP}{d\rho}\,, \tag{6}$$

in terms of the ratio $\rho_{\rm m}/\rho_{\rm r}$. What is the value of the sound speed at $t_{\rm eq}$?

6. CDM transfer function

The growing solution of the Mezaros equation is $\delta_{\rm m} \propto 2/3 + y$ where $y = a/a_{\rm eq}$. Use this to show that the transfer function can be approximated by

$$T(k) = C\left[1 + A\left(\frac{k}{k_{\rm eq}}\right) + B\left(\frac{k}{k_{\rm eq}}\right)^2\right]^{-1}\,, \tag{7}$$

where A, B and C are constants to be determined.

7. Manipulations of synchronous gauge equations of motion

Two of the synchronous gauge equations of motion are

$$k^2 H' = -\frac{2\omega_{\rm r}\theta_{\rm r}}{a^2}\,, \qquad \theta_{\rm r}' = \frac{1}{4}k^2\delta_{\rm r}\,. \tag{8}$$

Use these equations to show that

$$H'' + 2\frac{a'}{a}H' = -\frac{\omega_{\rm r}\delta_{\rm r}}{2a^2}\,. \tag{9}$$

8. Hot dark matter

If the three neutrinos have a mass hierarchy where $m_{\nu_{\rm e}} \gg m_{\nu_\mu} \gg m_{\nu_\tau}$, and the difference between the electron and muon neutrino masses satisfies $(\Delta m)^2 = 2.5 \times 10^{-3}$ eV^2, estimate the contribution of the massive neutrinos to the density of the universe. At what temperature will the electron neutrino become non-relativistic and what is the size of the horizon at this epoch?