EXAMPLES SHEET 3: INFLATION AND TOPOLOGICAL DEFECTS

1. Flatness problem

In the radiation dominated epoch $t < t_{\rm eq}$, show that $\Omega^{-1} - 1$ is proportional to the square of the scale factor. Using the inferred observational constraint $0 \leq 1 - \Omega_{\rm eq} \leq 10^{-3}$, extrapolate back to the Planck epoch $t \sim 10^{-43}$ s to demonstrate the extreme initial fine-tuning

$$0 \leq 1 - \Omega_{\rm pl} \leq 10^{-58}\,.$$

2. Comoving scales

(a) Derive the formula for the number of e-foldings that occur before the end of inflation (for a slowly rolling scalar field):

$$N(\phi) \approx 8\pi G \int_{\phi_{\rm end}}^{\phi} d\phi'\, V(\phi') \left[\frac{dV(\phi')}{d\phi'}\right]^{-1}$$

(b) Consider a wave of comoving wavelength 1 h^{-1} Mpc. When did it cross the particle horizon?

(c) Assume inflation ended with instantaneous reheating to $T_R = 10^{14}$ GeV. Equating the critical density in the radiation era to the energy density after re-heating (to solve the flatness problem) gives

$$\frac{3}{32\pi G}\frac{1}{t_R^2} = \frac{\pi^2}{30} g T_R^4$$

where $g \approx 100$ counts the number of degrees of freedom. Use this information to deduce the time at which re-heating occurs (t_R) and hence determine how many e-foldings from the end of inflation the 1 h^{-1} Mpc comoving wavelength leaves the horizon.

(d) For a scalar potential $V \propto \phi^4$ find the value of the scalar field at the instant when the 1 h^{-1} Mpc comoving wavelength left the horizon

3. Power Law Inflation

By differentiating the Friedmann equation in the case of a Universe whose evolution is dominated by a real scalar field, ϕ, show that

$$\dot{H} = -4\pi G\dot{\phi}^2.$$

Using this result, show that the equation of motion for a scalar field with potential

$$V(\phi) \propto \exp\left(\sqrt{\frac{16\pi}{p}}\frac{\phi}{m_{\rm pl}}\right)$$

has an exact solution corresponding to $a(t) \propto t^p$, i.e. show that there is a power law solution to the coupled scalar field and Friedmann equations without imposing the slow roll condition. Under what condition for p can this model solve the flatness problem?

4. Chaotic Inflation

For the inflaton potential $V(\phi) = \lambda\phi^n/(n m_{\rm pl}^{n-4})$, use the slow equations to describe the evolution of a small region which becomes dominated by vacuum energy, i.e. $V(\phi) \gg \dot{\phi}^2$ etc. Hence, find the quasi-exponential solutions

$$\phi(t)^{(4-n)/2} = \phi_{\rm start}^{(4-n)/2} + \frac{(4-n)t}{2}\left(\frac{n\lambda}{24\pi}\right)^{1/2} m_{\rm pl}^{(6-n)/2}\,, \qquad n \neq 4\,,$$

$$\phi(t) = \phi_{\rm start}\exp\left(-\sqrt{\lambda/6\pi}\,m_{\rm pl}\,t\right)\,, \qquad n = 4\,,$$

$$a(t) = a_{\rm start}\exp\left[\frac{4\pi}{n m_{\rm pl}^2}(\phi_{\rm start}^2 - \phi^2(t))\right]\,.$$

Determine the value of ϕ when inflation ends. Given $V(\phi_{\rm start}) \approx m_{\rm pl}^4$, find the overall expansion factor due to inflation. Taking $n = 4$ and assuming $g \approx 100$ and $\lambda \sim 10^{-14}$, estimate the reheat temperature T_R and time t_R (the duration of inflation).

5. Density fluctuations: Naive Argument

(a) By realising that quantum fluctuations mean that different regions reach the end of inflation at different times, give a simple argument for the formula $\mathcal{P}_S^{1/2} \sim H^2/\dot{\phi}$.

(b) For a $\lambda\phi^4$ potential and assuming 60 e-folds of inflation, show that normalising to the observed value $\mathcal{P}_S^{1/2} \sim 10^{-5}$ leads to $\lambda \sim 10^{-12}$. This smallness of λ is not natural and is known as the fine tuning problem of inflation.

(c) For a $\lambda\phi^4$ potential, determine the dependence of $\mathcal{P}_S$ on wavenumber given that the perturbations are generated when modes leave the Hubble radius, i.e. when $k = aH$. You may neglect the time dependence of H.

6. Defect instability (Derrick's theorem)

The energy of a localized static solution for a N-component real scalar field $\mathbf{\Phi}(r)$ in D-dimensions is

$$E = \int d^D\mathbf{x}\left[\tfrac{1}{2}(\nabla\mathbf{\Phi})^2 + V(\mathbf{\Phi})\right]\,,$$

where the integral over all space converges. By considering $E(\alpha)$ under the rescaling $\mathbf{x} \to \alpha\mathbf{x}$ and assuming that $V(\mathbf{\Phi})$ is non-negative, prove that there are no stable time-independent finite energy solutions in more than one dimension. Assume three dimensions and a spontaneously broken global symmetry $SO(N) \to SO(N-1)$ where $N > 2$, explain why the solutions are unstable ($N = 3$ are global monopoles and $N = 4$ are known as global textures). What happens if you introduces a fourth-order derivative term into the energy and set the potential to zero?

7. Sine-Gordon domain wall

(a) Consider a real scalar field ϕ described by a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - 2\lambda\eta^4\left[1 - \cos\left(\phi/\eta\right)\right] .$$

Show that the classical equation of motion is

$$\partial_\mu\partial^\mu\phi + 2\lambda\eta^3\sin(\phi/\eta) = 0$$

and that this equation admits a time-independent solution in one-dimension with $\phi(-\infty) = 0$ and $\phi(\infty) = 2\pi\eta$ of the form

$$\phi(x) = \alpha\tan^{-1}\left[\exp\left(\beta x\right)\right] ,$$

where α and β are coefficients to be determined.