REVIEW OF FRW UNIVERSE

Metric :

Global homogeneity and isotropy imply the FRW metric given here in terms its line element

$$\begin{aligned} ds^2 &= dt^2 - a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big] \\ &= a^2(\eta)\Big[d\eta^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big] \\ &= a^2(\eta)\Big[d\eta^2 - d\chi^2 - S_k^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\Big] \end{aligned}$$

where

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} \quad \rightarrow \quad S_k(\chi) = \sin\chi, \chi\,, \sinh\chi \text{ for } k = +1, 0, -1\,, \tag{1}$$

and

$$dt = a\,d\eta \quad \rightarrow \quad t = \int_0^\eta a(\eta')d\eta' \quad \text{and} \quad \eta = \int_0^t \frac{dt'}{a(t')}\,. \tag{2}$$

η is known as the conformal time. a is the scale factor. One can define it to have a particular value at one given time in the universe. Conventionally I will use $a(t_0) = a_0 = 1$ (t_0 is "now"). At the Big Bang singularity $t = 0$, $a = 0$ and $\eta = 0$.

Dynamics :

The Einstein equations imply

$$\left(\frac{\dot a}{a}\right)^2 = H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{1}{3}\Lambda\,, \quad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{1}{3}\Lambda\,. \tag{3}$$

We can define the critical density $\rho_{\rm crit}(t) = 3H^2/8\pi G$ and the present day critical density $\rho_{\rm crit}(t_0) = 3H_0^2/8\pi G$. When it is specified without a particular time, one is likely to be talking about the present day value. Densities of particular components are usually specified by reference to the present day critical density:

$$\Omega_{\rm X} = \frac{\rho_{\rm X}}{\rho_{\rm crit}}\,. \tag{4}$$

In this case the Friedmann equation can be re-written as

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\rm r}}{a^4} + \frac{\Omega_{\rm m}}{a^3} + \frac{\Omega_{\rm k}}{a^2} + \Omega_\Lambda\,, \tag{5}$$

where $\Omega_{\rm r} + \Omega_{\rm m} + \Omega_{\rm k} + \Omega_\Lambda = 1$, $\rho_\Lambda = -P_\Lambda = \Lambda/8\pi G$ and $\Omega_{\rm k} = -k/(H_0^2)$. We observe $\Omega_{\rm r} \sim 10^{-5}$ and it is usually ignored in this summation relation, but not in the Friedmann equation itself (since it is important at small enough a).

The age of the universe can be computed using

$$t_0 = \int_0^{t_0} dt' = H_0^{-1} \int_0^1 \frac{da}{\sqrt{\frac{\Omega_\mathrm{r}}{a^2} + \frac{\Omega_\mathrm{m}}{a} + \Omega_\mathrm{k} + \Omega_\Lambda a^2}} \,, \tag{6}$$

from which one can deduce that $H_0 t_0 = 2/3$ for $\Omega_\mathrm{m} = 1$ and $\Omega_\mathrm{r} = \Omega_\mathrm{k} = 0$. You might like to confirm that $H_0 t_0 \approx 1$ for $\Omega_\mathrm{m} = 0.25$ and $\Omega_\Lambda = 0.75$, which gives an age of 13.7 billion years.

Solutions :

By examining the dependence of each of the energy components with a one can see that as a increases from zero, the expansion of the universe is determined initially by radiation, then by matter and eventually by curvature and the cosmological constant if they exist. These four different epochs can be termed as radiation, matter, curvature and Λ domination, and the scale factor can be computed in each of the different eras:

$$\begin{aligned}
t < t_\mathrm{eq} \quad &: \quad a \propto t^{1/2} \propto \eta \\
t_\mathrm{eq} < t < t_\mathrm{curv} \quad &: \quad a \propto t^{2/3} \propto \eta^2 \\
t_\mathrm{curv} < t < t_\Lambda \quad &: \quad a \propto t \propto \exp\left[\sqrt{\Omega_\mathrm{k} H_0^2}\eta\right] \\
t > t_\Lambda \quad &: \quad a \propto \exp\left[\sqrt{\Omega_\Lambda H_0^2} t\right] \propto \left[1 + \sqrt{\Omega_\Lambda H_0^2}(\eta - \eta_0)\right]^{-1}
\end{aligned}$$

Distance measures :

The coordinate distance r is given by $r = S_k(\chi)$ where

$$\chi = \int_0^z \frac{dz'}{H(z')} \,. \tag{7}$$

It is the distance between us and an object at redshift z along a null geodesic. Observable distance measures can be written in terms of it.

The luminosity distance $d_\mathrm{L} = (1+z)r(z)$ is defined using the relationship between observed flux, F_o, and emitted luminosity, L_e,

$$F_\mathrm{o} = \frac{L_\mathrm{e}}{4\pi d_\mathrm{L}^2} \,. \tag{8}$$

The angular diameter distance $d_\mathrm{A} = r(z)/(1+z)$ relates the observed angular size of object θ and it is physical comoving size R,

$$R = d_\mathrm{A}\theta \,. \tag{9}$$