POWER SERIES SOLUTIONS

Eliminate h to give

$$
\begin{aligned}
\delta_{\rm m}' &= \frac{a}{a'}\left[k^2 H - \frac{3\omega_{\rm r}\delta_{\rm r}}{2a^2} - \frac{3\omega_{\rm m}\delta_{\rm m}}{2a}\right] \,, \\
\delta_{\rm r}' &= \frac{4}{3}\left(\delta_{\rm m}' - \theta_{\rm r}\right) \,, \\
\theta_{\rm r}' &= \frac{1}{4}k^2\delta_{\rm r} \,, \\
k^2 H' &= -\frac{2\omega_{\rm r}\theta_{\rm r}}{a^2} \,.
\end{aligned} \tag{1}
$$

Objective: to compute the first terms in possible regular power series solutions:

$$
\begin{aligned}
\delta_{\rm m} &= \sum_{n=0}^{\infty} c_n^{\rm m}\eta^n \,, \\
\delta_{\rm r} &= \sum_{n=0}^{\infty} c_n^{\rm r}\eta^n \,, \\
\theta_{\rm r} &= \sum_{n=0}^{\infty} c_n^{\theta}\eta^n \,, \\
H &= \sum_{n=0}^{\infty} c_n^{\rm H}\eta^n \,.
\end{aligned} \tag{2}
$$

Note that $a \approx \sqrt{\omega_{\rm r}}\eta + \omega_{\rm m}\eta^2/4$ and $a' \approx \sqrt{\omega_{\rm r}} + \omega_{\rm m}\eta/2$ since we are only considering only early times.

The last three of the equations of motion in (1) yield:

$$
c_1^{\rm r} + 2c_2^{\rm r}\eta + .. = \frac{4}{3}\left(c_1^{\rm m} + 2c_2^{\rm m}\eta - c_0^{\theta} - c_1^{\theta}\eta + ..\right) \,, \tag{3}
$$

$$
c_1^{\theta} + 2c_2^{\theta}\eta + 3c_3^{\theta}\eta^2 + .. = \frac{1}{4}k^2\left(c_0^{\rm r} + c_1^{\rm r}\eta + c_2^{\rm r}\eta^2 + ..\right) \,, \tag{4}
$$

$$
k^2\left(c_1^{\rm H} + 2c_2^{\rm H}\eta + ..\right) = -\frac{2}{\eta^2}\left(1 + \frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\theta} + c_1^{\theta}\eta + c_2^{\theta}\eta^2 + c_3^{\theta}\eta^3 + ..\right) \,. \tag{5}
$$

From these we can deduce that:

$$
c_1^{\rm r} = \frac{4}{3}(c_1^{\rm m} - c_0^{\theta}) \qquad c_2^{\rm r} = \frac{2}{3}(2c_2^{\rm m} - c_1^{\theta}) \,, \tag{6}
$$

$$
c_1^{\theta} = \frac{1}{4}k^2 c_0^{\rm r} \qquad c_2^{\theta} = \frac{1}{8}k^2 c_1^{\rm r} \qquad c_3^{\theta} = \frac{1}{12}k^2 c_2^{\rm r} \,, \tag{7}
$$

$$c_0^\theta = 0 \qquad c_1^\theta = 0 \qquad k^2 c_1^{\rm H} = -2c_2^\theta \qquad k^2 c_2^{\rm H} = -c_3^\theta + \frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}} c_2^\theta \,. \tag{8}$$

Hence we can deduce that $c_0^\theta = c_1^\theta = c_0^{\rm r} = 0$.

The first equation from (1) implies that:

$$\begin{aligned} c_1^{\rm m} + 2c_2^{\rm m}\eta + .. = & \quad k^2\eta\left(1+\frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}\eta\right)\left(1+\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\rm H}+c_1^{\rm H}\eta+..\right) \\ & - \frac{3}{2\eta}\left(1+\frac{\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(1+\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\rm r}+c_1^{\rm r}\eta+c_2^{\rm r}\eta^2+..\right) \\ & - \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\left(1+\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta\right)^{-1}\left(c_0^{\rm m}+c_1^{\rm m}\eta+..\right) \,. \end{aligned} \tag{9}$$

This confirms that $c_0^{\rm r} = 0$ and also that

$$c_1^{\rm m} = -\frac{3}{2}c_1^{\rm r} - \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}c_0^{\rm m} \,, \tag{10}$$

$$2c_2^{\rm m} = k^2 c_0^{\rm H} - \frac{3}{2}\left(c_2^{\rm r} - \frac{3\omega_{\rm m}}{4\sqrt{\omega_{\rm r}}}c_1^{\rm r}\right) - \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\left(c_1^{\rm m} - \frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}c_0^{\rm m}\right) . \tag{11}$$

Since $c_0^{\rm r} = c_0^\theta = 0$ there are two varieties of initial condition (i.e. behaviour at $\eta \to 0$): (i) $c_0^{\rm H} = 1$ and $c_0^{\rm m} = 0$ (ii) $c_0^{\rm H} = 0$ and $c_0^{\rm m} = 1$. A general initial configuration can be written as a linear superposition of these two cases.

Case (i)

$$c_1^{\rm m} = -\frac{3}{2}c_1^{\rm r} \qquad c_1^{\rm r} = \frac{4}{3}c_1^{\rm m} \quad \to \quad c_1^{\rm m} = c_1^{\rm r} = 0 \quad \to \quad c_2^\theta = c_1^{\rm H} = 0 \,. \tag{12}$$

$$2c_2^{\rm m} = k^2 - \frac{3}{2}c_2^{\rm r} \qquad c_2^{\rm r} = \frac{4}{3}c_2^{\rm m} \qquad c_3^\theta = \frac{1}{12}k^2 c_2^{\rm r} \,, \tag{13}$$

from which one can deduce that

$$c_2^{\rm m} = \frac{1}{4}k^2 \qquad c_2^{\rm r} = \frac{1}{3}k^2 \qquad c_3^\theta = \frac{1}{36}k^4 \qquad c_2^H = -\frac{1}{36}k^2 \;, \tag{14}$$

and hence

$$\begin{aligned} \delta_{\rm m} &= \frac{1}{4}k^2\eta^2 + \mathcal{O}(\eta^3) \,, \\ \delta_{\rm r} &= \frac{1}{3}k^2\eta^2 + \mathcal{O}(\eta^3) \,, \\ \theta_{\rm r} &= \frac{1}{36}k^4\eta^3 + \mathcal{O}(\eta^4) \,, \end{aligned}$$

$$H = 1 - \frac{1}{36}k^2\eta^2 + \mathcal{O}(\eta^3)\,. \tag{15}$$

Case (ii)

$$c_1^{\rm m} = -\frac{3}{2}c_1^{\rm r} - \frac{3\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}} \qquad c_1^{\rm r} = \frac{4}{3}c_1^{\rm m} \quad \rightarrow \quad c_1^{\rm m} = -\frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}} \qquad c_1^{\rm r} = -\frac{2\omega_{\rm m}}{3\sqrt{\omega_{\rm r}}}\,, \tag{16}$$

which implies that

$$c_2^{\theta} = -\frac{k^2\omega_{\rm m}}{12\sqrt{\omega_{\rm r}}} \qquad c_1^{\rm H} = \frac{\omega_{\rm m}}{6\sqrt{\omega_{\rm r}}}\,. \tag{17}$$

and hence

$$\begin{aligned}
\delta_{\rm m} &= 1 - \frac{\omega_{\rm m}}{2\sqrt{\omega_{\rm r}}}\eta + \mathcal{O}(\eta^2)\,,\\
\delta_{\rm r} &= -\frac{2\omega_{\rm m}}{3\sqrt{\omega_{\rm r}}}\eta + \mathcal{O}(\eta^2)\,,\\
\theta_{\rm r} &= -\frac{\omega_{\rm m}}{12\sqrt{\omega_{\rm r}}}k^2\eta^2 + \mathcal{O}(\eta^3)\,,\\
H &= \frac{\omega_{\rm m}}{6\sqrt{\omega_{\rm r}}}\eta + \mathcal{O}(\eta^2)\,.
\end{aligned} \tag{18}$$