1 EXAMPLES SHEET 1: SOLUTIONS

1. Differentiate the Friedmann equation:

$$\begin{aligned} 2H\dot{H} &= \frac{8\pi G}{3}\dot{\rho} + 2K\frac{\dot{a}}{a^3} = -8\pi GH(\rho+P) + \frac{2K}{a^2}H \\ \text{i.e.} \quad \dot{H} &= -4\pi G(\rho+P) + \frac{K}{a^2} \end{aligned}$$

and we also know that $\dot{H} = \ddot{a}/a - (\dot{a}/a)^2$ hence

$$\frac{\ddot{a}}{a} = -4\pi G(\rho+P) + \frac{8\pi G}{3}\rho = -\frac{4\pi G}{3}(\rho+3P).$$

2. Friedmann says

$$H^2 = H_0^2\left(\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3}\right)$$

and we know that $H = a'/a^2$ where the prime indicates differentiation with respect to conformal time. Thus

$$a'^2 = \omega_r + a\omega_m \quad \text{where} \quad \omega_r = \Omega_r H_0^2 \quad \text{etc.}$$

We can integrate this to get

$$\eta = \left[\frac{2}{\omega_m}(\omega_r + a\omega_m)^{1/2}\right]_0^a$$

which can be re-arranged to give the scale factor:

$$a = \sqrt{\omega_r}\eta + \frac{1}{4}\omega_m\eta^2.$$

3. See the Natural Units handout (on the course web site).
(a) $10^{15}M_\odot = (2\times10^{45}\text{ kg})\frac{1\text{ GeV}}{1.78\times10^{-27}\text{ kg}} = 1.1\times10^{72}$ GeV
(b) 500 μJy = $(500\times10^{-32}\text{ Jm}^{-2})\frac{1\text{ GeV}^3 \times(1.97\times10^{-16})^2\text{m}^2}{1.6\times10^{-10}\text{ J}} = 1.2\times10^{-51}$ GeV^3 (Note: Jy is the Jansky, which is a unit of flux.)
(c) 1000 km s^{-1} = $(10^6\text{ms}^{-1})\frac{1\text{ GeV}^0 \times(6.58\times10^{-25}\text{s})}{1.97\times10^{-16}\text{ m}} = 3.3\times10^{-3}$

Planck density is $\approx 1.6\times10^{57}$ GeV^4 and 1 GeV^4 = 1 GeV/(1 GeV)$^{-3}$ = $(1.78\times10^{-27}\text{ kg})/(1.97\times10^{-16}\text{ m})^3 = 2.3\times10^{20}$ kg m^{-3}

10^{32} GeV^2 = $10^{32}\times 1$ GeV/1 GeV^{-1} = $10^{32}\times1.78\times10^{27}$ kg/$(1.97\times10^{-16}\text{m}) = 9\times10^{20}$ kg m^{-1}.

4. For pressureless matter $\rho = \rho_0/a^3$ where ρ_0 is the current density, i.e.

$$\frac{H^2}{H_0^2} = \frac{1}{a^3} \implies a^{1/2}\dot{a} = H_0 \implies \int_0^1 a^{1/2}\text{d}a = H_0t_0 \implies H_0t_0 = \frac{2}{3}.$$

5. Friedmann says that

$$H = H_0(\Omega_m(1+z)^3 + \Omega_\Lambda)^{1/2} = H_0(1 + \frac{3}{2}\Omega_m z + O(z^2)).$$

Hence we can write

$$\begin{aligned} d_L(z) &= (1+z)\int_0^z \frac{\mathrm{d}z'}{H_0(1+\frac{3}{2}\Omega_m z' + O(z'^2))} \\ &= H_0^{-1}(1+z)\int_0^z \mathrm{d}z'\ (1 - \frac{3}{2}\Omega_m z' + O(z'^2)) \\ &= H_0^{-1}(1+z)(z - \frac{3}{4}\Omega_m z^2 + O(z^3)) \\ &= H_0^{-1}(z + z^2(1 - \frac{3}{4}\Omega_m) + O(z^3)). \end{aligned}$$

We can compare this to the defining expression for the deceleration parameter:

$$H_0 d_L = z + \frac{1}{2}(1-q_0)z^2 + \cdots$$

Whence we get

$$\begin{aligned} \frac{1}{2}(1-q_0) &= 1 - \frac{3}{4}\Omega_m \\ \text{i.e.}\quad q_0 &= -1 + \frac{3}{2}\Omega_m. \end{aligned}$$

6. We need the scale factor at matter-radiation equality, i.e.

$$\frac{\Omega_r}{a_{\rm eq}^4} = \frac{\Omega_m}{a_{\rm eq}^3} \implies a_{\rm eq} = \frac{\Omega_r}{\Omega_m}$$

from which we can deduce the conformal time using the result of Q.2:

$$a_{\rm eq} = H_0\sqrt{a_{\rm eq}\Omega_m}\eta_{\rm eq} + \frac{1}{4}H_0^2\Omega_m\eta_{\rm eq}^2.$$

Solving this quadratic gives the required answer. The comoving horizon at matter-radiation equality is just the conformal time $\eta_{\rm eq}$ and at the present epoch this is also the co-ordinate size of the horizon (since $a_0 = 1$). Putting $H_0^{-1} = 3000h^{-1}$ Mpc and $a_{\rm eq}^{-1} = 1 + z_{\rm eq} = 23980(\Omega_m h^2)$ (see the list of useful constants which accompanies the exam paper) gives $\eta_{\rm eq} \approx 2(\sqrt{2}-1)\Omega_m^{-1/2}\frac{3000h^{-1}}{\sqrt{23980(\Omega_m h^2)}}\text{Mpc} \approx 16(\Omega_m h^2)^{-1}\text{Mpc}$.

7. For the first part just use

$$\int \mathrm{d}\Omega\ Y_{lm}^*(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}.$$

Azimuthal symmetry means

$$a_{lm} = \int \frac{\Delta T}{T}(\theta) \sin\theta \; \mathrm{d}\theta \int_0^{2\pi} Y^*_{lm}(\theta, \phi) \mathrm{d}\phi.$$

Putting

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \mathrm{e}^{im\phi} P_l^m(\cos\theta)$$

we can do the integral over azimuth, which is non-zero only for $m = 0$. Hence the final answer:

$$a_{l0} = 2\pi \sqrt{\frac{2l+1}{4\pi}} \int_0^{\pi} \frac{\Delta T}{T}(\theta) P_l(\cos\theta) \sin\theta \; \mathrm{d}\theta.$$

8. The correlation function is defined by

$$\xi(\mathbf{r}) = \frac{1}{V} \int \mathrm{d}^3\mathbf{x} \; \Delta(\mathbf{x} + \mathbf{r}) \Delta^*(\mathbf{x}),$$

which reduces to

$$\xi(\mathbf{r}) = \frac{1}{V} \sum_{ij} \int \mathrm{d}^3\mathbf{x} \; \delta(\mathbf{x} + \mathbf{r} - \mathbf{x}_i) \delta(\mathbf{x} - \mathbf{x}_j)$$

in the particular case we are studying. The result follows after integrating over $\mathbf{x}$. To obtain the power spectrum we need to remember that

$$\Delta(\mathbf{x}) = \frac{V}{(2\pi)^3} \sum_{\mathbf{k}} \Delta_{\mathbf{k}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}},$$

from which we can show that

$$\xi(\mathbf{r}) = \frac{V}{(2\pi)^3} \sum_{\mathbf{k}} |\Delta_{\mathbf{k}}|^2 \, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}}.$$

The power spectrum is thus the Fourier transform of the correlation function:

$$|\Delta_{\mathbf{k}}|^2 = \frac{1}{V} \int \mathrm{d}^3\mathbf{r} \; \xi(\mathbf{r}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{V^2} \sum_{ij} \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}.$$

9. In the radiation era $a \propto t^{1/2}$ and so $H = \frac{1}{2t}$. Likewise, in the matter era $a \propto t^{2/3}$ so $H = \frac{2}{3t}$. Thus the critical density is $\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G} = \frac{3}{32\pi G t^2}$ in the radiation era and $\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G} = \frac{3}{6\pi G t^2}$ in the matter era.

10. In the radiation era, $H^2 = (8\pi G/3)\rho_r$ where $\rho_r = (\pi^2/30) g T^4$. From the previous question we also know that $H = 1/(2t)$ in this era. Thus

$$\frac{1}{4t^2} = \frac{8\pi G}{3} \frac{\pi^2}{30} g T^4.$$

Re-arranging and putting $m_{\mathrm{pl}}^2 = 1/G$ gives

$$t = \sqrt{\frac{90}{32\pi^3}} g^{-1/2} \frac{m_{\mathrm{pl}}}{T^2} \approx 0.3 g^{-1/2} \frac{m_{\mathrm{pl}}}{T^2}.$$