1 EXAMPLES SHEET 2: SOLUTIONS

1. Correlation function is just

$$\begin{aligned}
\xi(\mathbf{r}) &= \frac{1}{V}\int d^3\mathbf{x}\ \Delta(\mathbf{x}+\mathbf{r})\Delta^*(\mathbf{x}) \\
&= \frac{1}{V}\sum_{ij}\int d^3\mathbf{x}\ \delta(\mathbf{x}+\mathbf{r}-\mathbf{x}_i)\delta(\mathbf{x}-\mathbf{x}_j) \\
&= \frac{1}{V}\sum_{ij}\ \delta(\mathbf{r}-\mathbf{x}_i+\mathbf{x}_j).
\end{aligned}$$

To deduce the power spectrum we need

$$|\Delta_k|^2 = \frac{1}{V}\int d^3\mathbf{r}\ \xi(\mathbf{r})\ \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{V^2}\sum_{ij}\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}.$$

2. Putting $\delta_m = (2/3+y)v(y)$ as suggested gives

$$\left(\frac{2}{3}+y\right)\frac{\mathrm{d}^2v}{\mathrm{d}y^2} + \left[2+\frac{\frac{3}{2}(\frac{2}{3}+y)^2}{y(1+y)}\right]\frac{\mathrm{d}v}{\mathrm{d}y} = 0,$$

which can be re-written as

$$\frac{\mathrm{d}^2v}{\mathrm{d}y^2} = -\left[\frac{6}{2+3y}+\frac{1}{y}+\frac{1}{2(1+y)}\right]\frac{\mathrm{d}v}{\mathrm{d}y}.$$

Integrating gives

$$\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{C}{y(2+3y)^2(1+y)^{1/2}}$$

and integrating again gives

$$v = C\int\frac{\mathrm{d}y}{y(2+3y)^2(1+y)^{1/2}}.$$

For $y \ll 1$ this reduces to

$$v \approx \frac{C}{4}\int\frac{\mathrm{d}y}{y} = \frac{C}{4}\ln y \implies \delta_m \propto \ln y$$

and for $y \gg 1$ it is

$$v \approx \frac{D}{9}\int\frac{\mathrm{d}y}{y^{7/2}} = -\frac{2D}{45}y^{-5/2} \implies \delta_m \propto y^{-3/2}.$$

In the lectures we found $\delta_m \propto (\ln\eta + C)$ in the radiation era ($y \ll 1$) and $\delta_m \propto At^{2/3} + B/t$ in the matter era ($y \gg 1$). Since $a \propto t^{2/3}$ in the matter era and $a \propto \eta$ in the radiation era agreement follows.

3. We want to compute

$$\sigma_R^2 = \frac{1}{V}\int \mathrm{d}^3\mathbf{x}\ |\delta(R,\mathbf{x})|^2$$

where

$$\begin{aligned}
|\delta(R,\mathbf{x})|^2 &= \frac{1}{V^2}\int |W(\mathbf{x}'-\mathbf{x})\delta(\mathbf{x}')|\ |W(\mathbf{x}''-\mathbf{x})\delta(\mathbf{x}'')|\ \mathrm{d}^3\mathbf{x}'\ \mathrm{d}^3\mathbf{x}'' \\
&= \frac{1}{V^2}\int W(\mathbf{x}'-\mathbf{x})W(\mathbf{x}'+\mathbf{r}-\mathbf{x})\delta(\mathbf{x}')\delta(\mathbf{x}'+\mathbf{r})\ \mathrm{d}^3\mathbf{x}'\ \mathrm{d}^3\mathbf{r} \\
&= \frac{1}{V^2}\frac{V^2}{(2\pi)^6}\sum_{\mathbf{k},\mathbf{k}'}\int W(\mathbf{x}'-\mathbf{x})W(\mathbf{x}'+\mathbf{r}-\mathbf{x})\delta_{\mathbf{k}}\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}'}\delta_{\mathbf{k}'}\ \mathrm{e}^{-i\mathbf{k}'\cdot(\mathbf{x}'+\mathbf{r})}\mathrm{d}^3\mathbf{x}'\ \mathrm{d}^3\mathbf{r}.
\end{aligned}$$

Now let $\mathbf{x}'-\mathbf{x}=\mathbf{u}$

$$\begin{aligned}
|\delta(R,\mathbf{x})|^2 &= \frac{1}{(2\pi)^6}\sum_{\mathbf{k},\mathbf{k}'}\int W(\mathbf{u})W(\mathbf{u}+\mathbf{r})\delta_{\mathbf{k}}\delta_{\mathbf{k}'}\ \mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{x}+\mathbf{u})}\ \mathrm{e}^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{u}+\mathbf{r})}\mathrm{d}^3\mathbf{u}\ \mathrm{d}^3\mathbf{r}. \\
&= \frac{1}{(2\pi)^6}\sum_{\mathbf{k},\mathbf{k}'} W_{\mathbf{k}}W_{\mathbf{k}'}\ \delta_{\mathbf{k}}\delta_{\mathbf{k}'}\ \mathrm{e}^{-i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sigma_R^2 &= \frac{1}{(2\pi)^3}\sum_{\mathbf{k}} W_{\mathbf{k}}W_{-\mathbf{k}}\ \delta_{\mathbf{k}}\delta_{-\mathbf{k}} \\
&= \frac{1}{(2\pi)^3}\sum_{\mathbf{k}} |W_{\mathbf{k}}|^2\ \ |\delta_{\mathbf{k}}|^2
\end{aligned}$$

and hence the result, since $P_{\mathbf{k}} = |\delta_{\mathbf{k}}|^2$.

For $P_i = Ak^n$ we have (whilst outside the horizon) $P(k,\eta) = (\eta/\eta_i)^4 P_i$ and thus

$$\begin{aligned}
\sigma_R^2 &= \frac{A}{(2\pi)^3}\frac{\eta^4}{\eta_i^4}\int \mathrm{d}^3\mathbf{k}\ |W(kR)|^2\ k^n \\
&= \frac{4\pi A}{(2\pi)^3}\frac{\eta^4}{\eta_i^4}\int_0^\infty k^{n+2}|W(kR)|^2\mathrm{d}k \\
&= \frac{A}{2\pi^2}\frac{\eta^4}{\eta_i^4}R^{-n-3}\int_0^\infty \mathrm{d}x\ x^{n+2}\,|W(x)|^2 \\
&\propto \eta^{1-n} \quad \text{for} \quad \eta = R,
\end{aligned}$$

which is independent of time if $n=1$.

4. We can approximate the integral over k as follows:

$$\sigma_R^2 \approx \int_0^{k_{\mathrm{eq}}} \frac{\mathrm{d}k}{(2\pi)^3}\ B\frac{k}{k_{\mathrm{eq}}^4}4\pi k^2 W(kR)^2 + \int_{k_{\mathrm{eq}}}^{1/R} \frac{\mathrm{d}k}{(2\pi)^3}\ A\frac{1}{k^3}4\pi k^2 W(kR)^2$$

and have assumed the window function cuts off sharply at $kR > 1$. Matching the two integrands at $k_{\rm eq}$ fixes $B \approx A$ and converting to dimensionless integrals:

$$\sigma_R^2 \approx \int_0^{k_{\rm eq}R} \frac{{\rm d}y}{4\pi^2}\ A\frac{y^3}{(Rk_{\rm eq})^4}W(y)^2 + \int_{k_{\rm eq}R}^1 \frac{{\rm d}y}{4\pi^2}\ A\frac{1}{y}W(y)^2.$$

For $k_{\rm eq}R < 1$, the second term dominates (except very close to $k_{\rm eq}R = 1$) and so we can fix A given the window function and $\sigma_8 = 0.8$. For $k_{\rm eq}R \gg 1$ it is the first term that matters, i.e.

$$\sigma_R^2 \approx \frac{1}{(Rk_{\rm eq})^4}\int_0^{k_{\rm eq}R} \frac{{\rm d}y}{4\pi^2}\ Ay^3W(y)^2 \sim A.$$

So a measurement of σ_8 allows us to predict σ_R for $R \gg k_{\rm eq}^{-1}$.
If the dark matter is hot, we know that free-streaming suppresses power on small scales, i.e. $P(k) \sim 0$ for $k > k_{FS}$ where $k_{FS}^{-1} \sim (1-10)$ Mpc.

5. We know $P = \rho_r/3$ and $\rho = \rho_m + \rho_r$. We also know that

$$\frac{{\rm d}\rho}{{\rm d}a} = -\frac{(\rho_m + \frac{4}{3}\rho_r)}{a}.$$

Thus

$$\frac{{\rm d}P}{{\rm d}\rho} = \frac{{\rm d}P}{{\rm d}a}\frac{{\rm d}a}{{\rm d}\rho} = \frac{-\frac{4}{9}\frac{\rho_r}{a}}{-\frac{(\rho_m+\frac{4}{3}\rho_r)}{a}}$$

and hence

$$c_s^2 = \frac{1}{3}\left(1 + \frac{3}{4}\frac{\rho_m}{\rho_r}\right)^{-1}.$$

At t_{eq}, $c_s^2 = \frac{1}{3}\left(1+\frac{3}{4}\right)^{-1} = \frac{4}{21}$.

6. We know that the modes grow until they cross the horizon at $\eta_H = 2\pi/k$ and that they stop growing at η_Λ. In the intervening period they experience a Mezaros growth. Thus

$$\delta_m(\eta_0) = \left(\frac{\eta_H}{\eta_i}\right)^2 \frac{\frac{2}{3} + \frac{a_\Lambda}{a_{eq}}}{\frac{2}{3} + \frac{a_H}{a_{eq}}}\delta_m(\eta_i)$$

and so

$$T(k) = \left(\frac{\eta_H}{\eta_i}\right)^2 \frac{\frac{2}{3} + \frac{a_\Lambda}{a_{eq}}}{\frac{2}{3} + \frac{a_H}{a_{eq}}}.$$

We need to compute the various terms. For a matter-radiation universe we can use the Friedmann equation ($a' = \sqrt{\omega_m a + \omega_r}$ with $\omega_i = \Omega_i H_0^2$) to derive that

$$\begin{aligned} a &= \sqrt{\omega_r}\eta + \frac{1}{4}\omega_m\eta^2 \quad \text{and} \\ \eta &= \frac{2}{\omega_m}\left[(\omega_r + \omega_m a)^{1/2} - \omega_r^{1/2}\right]. \end{aligned}$$

Since $a_{eq} = \omega_r/\omega_m$ this gives

$$\begin{aligned}
\eta_{eq} &= 2(\sqrt{2}-1)\frac{\omega_r^{1/2}}{\omega_m} \\
\therefore \frac{a}{a_{eq}} &= 2(\sqrt{2}-1)\frac{\eta}{\eta_{eq}} + (3-2\sqrt{2})\left(\frac{\eta}{\eta_{eq}}\right)^2.
\end{aligned}$$

Substituting in gives

$$\begin{aligned}
T(k) &= \left(\frac{2}{3}+\frac{a_\Lambda}{a_{eq}}\right)\left(\frac{\eta_{eq}}{\eta_i}\right)^2 \frac{(\eta_H/\eta_{eq})^2}{\frac{2}{3}+2(\sqrt{2}-1)\frac{\eta_H}{\eta_{eq}}+(3-2\sqrt{2})\left(\frac{\eta_H}{\eta_{eq}}\right)^2} \\
&= \left(\frac{2}{3}+\frac{a_\Lambda}{a_{eq}}\right)\left(\frac{\eta_{eq}}{\eta_i}\right)^2 (3-2\sqrt{2})^{-1}\left[\frac{2}{3}\frac{1}{(3-2\sqrt{2})}\left(\frac{\eta_{eq}}{\eta_H}\right)^2 + \frac{2(\sqrt{2}-1)}{(3-2\sqrt{2})}\frac{\eta_H}{\eta_{eq}}+1\right]^{-1}
\end{aligned}$$

which is of the required form upon writing $\eta_{eq}/\eta_H = k/k_{eq}$.

7. Differentiate the first equation to get

$$\begin{aligned}
k^2 H'' &= -2\omega_r\left[\frac{\theta_r'}{a^2} - \frac{2a'}{a^3}\theta_r\right] \\
&= -\frac{2\omega_r}{a^2}\frac{1}{4}k^2\delta_r - \frac{2a'}{a}k^2 H' \\
\Longrightarrow k^2\left(H''+\frac{2a'}{a}H'\right) &= -\frac{\omega_r}{2a^2}k^2\delta_r.
\end{aligned}$$

8. Data give $m_{\nu_e} = \sqrt{2.5\times 10^{-3}}\text{eV} = 0.05$ eV. From the notes we know that

$$\Omega_\nu h^2 \approx \frac{m_{\nu_e}}{94 \text{ eV}} \approx 5.3\times 10^{-4}.$$

The electron-neutrino becomes non-relativistic when $T \approx m = 0.05$ eV. Using $a \propto 1/T$ gives

$$\begin{aligned}
\frac{a_0}{a_{NR}} &= \frac{T_{NR}}{T_0} = \frac{0.05 \text{ eV}}{2.728 \text{ K}} \\
&= \frac{0.05 \text{ eV}}{2.728 \text{ K}} 1.16\times 10^4 \text{ (K/eV)} \\
&= 210.
\end{aligned}$$

To get the horizon size use

$$d_H(t_{NR}) = \frac{3}{2}t_{NR} = \frac{3}{2}t_0\left(\frac{a_{NR}}{a_0}\right)^{3/2}.$$