1 EXAMPLES SHEET 3: SOLUTIONS

1. From the notes we have (i.e. take the Friedmann equation in the form $\Omega - 1 = K/(aH)^2$ and differentiate w.r.t. time, then use the Raychaudhuri equation for $\ddot{a}/a$)

$$\Omega^{-1} - 1 = \frac{\Omega_0 - 1}{\Omega_0} a^{1+3w}$$

which reduces to the required answer with $w = 1/3$. The extrapolation back to the Planck epoch follows upon realising that $a^2 \propto t$ in the radiation era and putting $t_{\rm eq} \sim 10^{12}$ s.

2. (a) See lecture notes, i.e. use $H = \frac{\mathrm{d}a}{\mathrm{d}\phi}\dot{\phi}/a$ together with the equation $H/\dot{\phi} = -8\pi G(V/V')$, which arises from the slow roll equations.
(b) The comoving wavelength is smaller than the comoving horizon size at $t_{\rm eq}$ (which is around $16(\Omega_m h^2)^{-1}$ Mpc from Q6 of sheet 1) so it crossed the horizon in the radiation era when $a \propto \eta \propto \sqrt{t}$ and hence we can write

$$\frac{a}{a_{\rm eq}} = \frac{\eta}{\eta_{\rm eq}} = \frac{1\ h^{-1}}{16(\Omega_m h^2)^{-1}} = \sqrt{\frac{t_H}{t_{\rm eq}}}.$$

Putting $t_{\rm eq} = 3.15 \times 10^{10}\ (\Omega_m h^2)^{-2}$ s, $\Omega_m = 0.3$ and $h = 0.7$ gives $t_H = 3 \times 10^8$ s. You can get the same answer by thinking in terms of the physical wavelength, i.e.

$$\lambda(t_H) = \frac{a(t_H)}{a_0}\lambda(t_0) = \frac{a(t_H)}{a_{\rm eq}}\frac{a_{\rm eq}}{a_0}\lambda(t_0) = 2t_H.$$

The final equality uses the fact that the particle horizon during the radiation era is equal to $2t$. Re-arranging, and remembering that the comoving wavelength is equal to the physical wavelength at the present epoch, gives

$$t_H = \frac{1}{2}\left(\frac{t_H}{t_{\rm eq}}\right)^{1/2} \frac{1\ h^{-1}\ \mathrm{Mpc}}{1+z_{\rm eq}} \implies t_H = \frac{(1\ h^{-1}\ \mathrm{Mpc})^2}{4t_{\rm eq}(1+z_{\rm eq})^2}$$

which gives the same answer as before after substituting $z_{\rm eq} = 23980(\Omega_m h^2)$ and 1 Mpc $= 1.0 \times 10^{14}$ s.
(c) Re-heating (i.e. end of inflation) occurs at

$$t_R = \left(\frac{90 m_{\rm pl}^2}{32\pi^3 g}\right)^{1/2} \frac{1}{T_R^2} = 2 \times 10^{-35}\ \mathrm{s}.$$

The number of e-foldings is (using the result in part (b) for t_H)

$$e^N = \left(\frac{t_H}{t_R}\right)^{1/2} \implies N = \frac{1}{2}\log\left(\frac{t_H}{t_R}\right) \approx 50$$

(d) Using the result of part (a) gives

$$N = \frac{8\pi}{m_{\rm pl}^2}\int_{\phi_{\rm end}}^{\phi} {\rm d}\phi' \, \frac{\phi'}{4} = \pi\frac{\phi^2 - \phi_{\rm end}^2}{m_{\rm pl}^2}.$$

Hence, using the result of part (c) and assuming $\phi_{\rm end} \ll \phi_{50}$:

$$50 \approx \pi\left(\frac{\phi_{50}}{m_{\rm pl}^2}\right)^2 \implies \phi_{50} \approx 4m_{\rm pl}.$$

3. Differentiating the Friedmann gives $2H\dot{H} = (8\pi G/3)(\dot{V}+\dot{\phi}\ddot{\phi}) = -8\pi GH\dot{\phi}^2$ where we used the equation of motion ($\ddot{\phi} + 3H\dot{\phi} + V' = 0$) to substitute for $\ddot{\phi}$. Assuming that $a \propto t^p$ is a solution we shall attempt to derive the corresponding potential. We have $H = p/t$ and $\dot{H} = -p/t^2$. Hence $\dot{\phi} = -(p/4\pi G)^{1/2}/t$ (want minus sign since field rolls down the potential) and $\ddot{\phi} = +(p/4\pi G)^{1/2}/t^2$. The form of the potential suggests that we evaluate

$$\frac{V'}{V} = \frac{-\ddot{\phi} - 3H\dot{\phi}}{3H^2/(8\pi G) - \dot{\phi}^2/2} = \sqrt{\frac{16\pi}{p}}\frac{1}{m_{\rm pl}} \implies V(\phi) = \exp\left(\int^{\phi} {\rm d}\phi \, \sqrt{\frac{16\pi}{p}}\frac{1}{m_{\rm pl}}\right).$$

The flatness problem can be solved if $\ddot{a} > 0$ which implies that $p > 1$.

4. Use $H^2 = (8\pi G/3)V$ and $3H\dot{\phi} + V' = 0$ to establish that

$$\dot{\phi} = -\left(\frac{n\lambda}{24\pi}\right)^{1/2} m_{\rm pl}^{3-n/2}\phi^{n/2-1}$$

which integrates to the required solutions for $n = 4$ and $n \neq 4$. To get the evolution of the scale factor use ${\rm d}a/{\rm d}\phi = aH/\dot{\phi}$ and $H/\dot{\phi} = -8\pi GV/V'$ from the slow roll equations.Inflation ends when $\dot{\phi}^2 = V$ which leads to $\phi_{\rm end} = nm_{\rm pl}/\sqrt{24\pi}$. To get the expansion factor use the equation for the scale factor, i.e.

$$\frac{a_{\rm end}}{a_{\rm start}} = \exp\left(-\frac{n}{6} + \frac{4\pi}{n}\left(\frac{n}{\lambda}\right)^{2/n}\right)$$

To get T_R we simply need

$$\frac{\pi^2}{30}gT_R^4 = V(\phi_{\rm end}) = \frac{\lambda}{4}\left(\frac{4m_{\rm pl}}{\sqrt{24\pi}}\right)^4 \implies T_R \approx \left(\frac{\lambda}{30}\right)^{1/4}\frac{m_{\rm pl}}{\pi} \approx 4{\times}10^{-5}m_{\rm pl}.$$

And to get t_R we can solve

$$\phi_{\rm end} = \phi_{\rm start}\exp\left(-\left(\frac{10^{-14}}{6\pi}\right)^{1/2} m_{\rm pl}t_R\right) \implies t_R \approx 4\times 10^8 \; m_{\rm pl}^{-1}.$$

5. (a) Field rolls down the potential with fluctuations $\delta\phi \sim H$ in the scalar field due to the Gibbons-Hawking effect. When the field reaches $\phi_{\rm end}$ reheating takes place but since the precise value of ϕ is uncertain this will occur at different times in different regions with $\delta t = \delta\phi/\dot\phi$. The scalar power spectrum is therefore

$$P_S^{1/2} = \frac{\delta\rho}{\rho} \sim \frac{\delta a}{a} \sim H\delta t \sim \frac{H^2}{\dot\phi}.$$

(b) Using the previous result:

$$\frac{H^2}{\dot\phi} \sim \frac{1}{m_{\rm pl}^3}\frac{V^{3/2}}{V'} \sim \frac{\sqrt{\lambda}}{m_{\rm pl}^3}\phi^3 \sim 10^{-5}$$

which gives the required result upon putting $\phi = \phi_{60} = 4m_{\rm pl}$ (do not worry about factors of 4).

(c) For modes that are just exiting the Hubble radius

$$\begin{aligned} k &= Ha \propto \exp\left(-\phi^2\pi/m_{\rm pl}^2\right) \\ \therefore &\quad \log k^{-1} \sim \phi^2/m_{\rm pl}^2 \end{aligned}$$

and from part (b) we know that $P_S^{1/2} \sim (\phi/m_{\rm pl})^3$ therefore $P_S^{1/2} \sim (\log k^{-1})^{3/2}$ (we have treated H as a constant, which you should confirm is the case).

6. After re-scaling

$$E(\alpha) = \alpha^{D-2}I_K + \alpha^D I_V$$

where

$$I_K = \frac{1}{2}\int d^D\mathbf{x}\ |\boldsymbol{\nabla}\boldsymbol{\Phi}|^2 \quad \text{and } I_V = \int d^D\mathbf{x}\ V(\boldsymbol{\Phi}).$$

So we see that it is possible to reduce the energy to arbitarily small values by reducing α and this is inconsistent with there being stable finite-energy solution with $\alpha = 1$. Note that for $D = 2$, this argument only works if $I_V \neq 0$. The preceding argument means that there can be no monopoles or textures in this theory, even though they are allowed on topological grounds. This is an illustration of the fact that the topological condition for the existence of defects is a necessary but not sufficient condition for their existence. Adding a term

$$I_4 = \frac{1}{2}\int d^D\mathbf{x}\ |\boldsymbol{\nabla}\boldsymbol{\Phi}|^4 \quad \text{and putting} \quad I_V = 0$$

leads to a total energy

$$E(\alpha) = \alpha I_K + \alpha^{-1}I_4$$

for $D = 3$. There is a tension between the two terms now and stable solution exist for which $dE/d\alpha = 0$, i.e. $\alpha^2 = I_4/I_K$.

7. Use the Euler-Lagrange equations, i.e.

$$\frac{\partial L}{\partial(\partial_\mu \phi)} = \partial_\mu \phi \quad \text{and} \quad \frac{\partial L}{\partial \phi} = -2\lambda\eta^3 \sin(\phi/\eta)$$

and the result follows since

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = \frac{\partial L}{\partial \phi}.$$

In one-dimension, the equation of motion for time-independent solutions is

$$\frac{d^2\phi}{dx^2} = 2\lambda\eta^3 \sin(\phi/\eta).$$

The given solution satisfies $\phi(+\infty) = \alpha\pi/2$ and so $\alpha = 4\eta$. To prove it is a solution you will need to show that

$$\frac{d^2\phi}{dx^2} = -2\eta\beta^2 \frac{\sinh(\beta x)}{\cosh^2(\beta x)}$$

and

$$\begin{aligned}
\sin(\phi/\eta) &= 2\sin(\phi/2\eta)\cos(\phi/2\eta) \\
&= 2\frac{2\tan(\phi/4\eta)}{1+\tan^2(\phi/4\eta)}\frac{1-\tan^2(\phi/4\eta)}{1+\tan^2(\phi/4\eta)} \\
&= \frac{4e^{\beta x}}{1+e^{2\beta x}}\frac{1-e^{2\beta x}}{1+e^{2\beta x}} \\
&= -2\frac{\sinh(\beta x)}{\cosh^2(\beta x)}.
\end{aligned}$$

Note we used the fact that $e^{\beta x} = \tan(\phi/4\eta)$. It follows that $\beta = \eta\sqrt{2\lambda}$.