Understanding Gluon Radiation In Interjet Regions

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Abstract

The "gap between jets" cross-section is a well studied example of a non-global observable. The non-global nature of this observable is reflected in the miscancellation of soft virtual and real gluon radiation corrections arising from primary and secondary "in gap" virtual gluons and throws light on the underlying Quantum Chromodynamic (QCD) processes at work. Due to the complexity of the underlying QCD processes, these effects have been most easily studied in the "large N_c" approximation, as manifest in the Banfi-Marchesini-Smye (BMS) evolution equation. This thesis presents work aimed at calculating the corrections to the Born cross-section for the process of electron-positron annihilation to form a quark-antiquark pair, keeping the full $N = 3$ dependence. The corrections arising from both primary and secondary real-virtual miscancellations are calculated for zero, one and two gluons outside the gap. It is intended, eventually, to use the results of these calculations to provide a measure of the accuracy of the BMS equation.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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The Author

The author is a Physician who has studied Physics part-time and in 2006 gained a first class BSc degree in Physics at the Open University.

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Chapter 1

Introduction

The process of electron-positron annihilation to form a quark-antiquark pair ($e^- e^+ \to q\overline{q}$) can be represented as the production of two outgoing jets (one quark jet and one antiquark jet) of co-moving particles separated by a rapidity gap Y. If the jets are of a transverse momentum (k_T) scale Q (the hard scale), it is possible to define a "gap between jets" such that no third jet with transverse momentum greater than a maximum transverse momentum Q_0 (the veto scale, where $Q \gg Q_0$) exists within the gap $[1, 2, 3]$.

Though the incoming electron and positron are "colourless" the quark-antiquark pair carry colour charge and thus give rise to real gluon radiation (bremsstrahlung). The nature of the gap definition means that any of this radiation, with a transverse momentum greater than Q_0, cannot be found within the gap. Both the quark, antiquark and any subsequent real gluon bremsstrahlung may also be dressed with virtual gluon corrections which being virtual may be present within the gap. The gap between jets observable is a "non-global" observable in the sense that it is sensitive to radiation only within a restricted part of phase space [4, 6]. The presence of virtual gluons and the absence of real gluons, with transverse momentum greater than Q_0, within the gap give rise to corrections to the Born cross-section, as explained below.

If k^μ is the four momentum of a radiated gluon and p^μ is the four momentum of the parent parton then a "soft" gluon is one for which all $k^\mu \ll$ maximum p^μ. Corrections due to soft gluons can be accurately described using the "eikonal" rules (as described below) [5]. Eikonal comes from the Greek $\varepsilon\iota\kappa\varepsilon\nu\alpha\iota$ to "resemble" (i.e. each eikonal gluon added is a copy (albeit with lower energy) of the previous one and the lowest order kinematics are left unchanged). The eikonal rules give rise to logarithmic corrections of the form $\ln\left(\frac{Q}{Q_0}\right)$ to the cross-section [5]. It is only soft gluons that give rise to logarithmic corrections of this form (hence the utility of the eikonal approximation).

Exponentiation of $\alpha_s \ln\left(\frac{Q}{Q_0}\right)$ (where α_s is the strong coupling constant) then describes the addition of any number of virtual loop corrections at markedly decreasing transverse momenta (the strong ordering approximation) such that $Q > k_{1T} > k_{2T} ..... > Q_0$ [6]. The gap cross-section σ may then generally be expressed as a perturbative expansion (also termed a "resummation) in the strong coupling constant of the logarithms of the hard scale Q to the cut-off scale Q_0 i.e.

$$\sigma = \sigma_0 \left(1 + \alpha_s \ln\left(\frac{Q}{Q_0}\right) + \alpha_s^2 \ln^2\left(\frac{Q}{Q_0}\right) + ...\alpha_s^n \ln^n\left(\frac{Q}{Q_0}\right)\right) \tag{1.1}$$

where the logarithms reflect the contribution of the uncancelled virtual corrections and σ_0 is the Born cross-section. This is the "leading logarithmic approximation" where leading refers to the powers of the logarithms being the same as that of the strong coupling constant. The eikonal approximation is thus sufficient to encompass the leading logarithmic corrections. The use of the eikonal approximation to calculate amplitudes produces expressions of the form:

$$\int d^4k \frac{p_3 \cdot p_4}{(2p_3 \cdot k \pm i\varepsilon)\,(2p_4 \cdot k \pm i\varepsilon)\,(k^2 + i\varepsilon)} \tag{1.2}$$

(where p_3, p_4 and k are the four momenta of the antiquark, quark and virtual gluon respectively). By contour integration over energy, the dimensionality of the loop integral may be reduced by one. This integration produces both a real ($\int d^3\mathbf{k}$) and a complex ($i\pi \int d^2\mathbf{k}$) integral. The real integral (of the virtual gluon) is associated with the exchanged gluon going on shell; this is termed the eikonal gluon contribution (eikonal is here used in a different sense from that applied to the soft corrections). The imaginary part results from the quark/antiquark propagators going on shell; this is the Coulomb gluon contribution [6]. For the $e^+e^- \to q\bar{q} + ng$ cross-section the complex $(i\pi)$ terms produce only an unimportant change of phase [8] and thus can safely be omitted in this work.

The Bloch Nordsieck Theorem (BNT) states that (any number of) soft (real and virtual) gluon corrections to the cross-section cancel exactly for an all inclusive observable (i.e. in this case the unrestricted four momentum phase space). This is sometimes expressed as the "sum over cuts" of geometrically equivalent diagrams is zero (see Figure 1.1 where the diagrams represent cross-sections for a real gluon emission in the upper two panes and a virtual emission in the lower two panes, from quark and antiquark lines. The black dots represent the the photon-

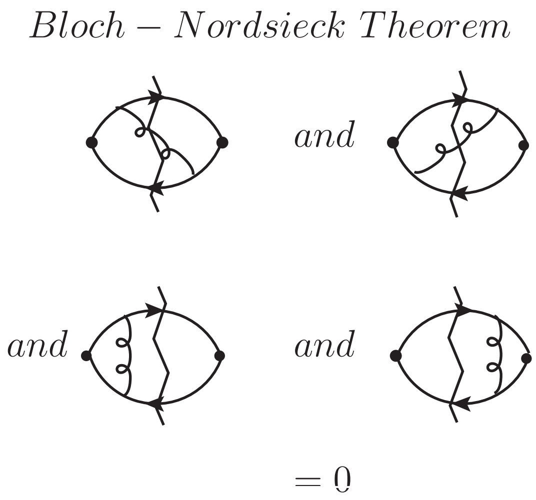

Figure 1.1: Bloch-Nordieck Cancellation

quark-antiquark vertices. The curved lines represent the quark and antiquark with the arrows indicating fermion flow, whilst the looped line represents a gluon (the electron, positron and photon lines are not shown). The the wavy vertical line represents the cut between the two components of the inner product).

If a cut-off in transverse momentum is introduced whereby no radiation within the gap is allowed with transverse momentum above the cut-off (Q_0), then a mis-cancellation in the BNT is introduced. This is due to the presence of virtual gluons in the gap with transverse momenta above Q_0 that have no real gluons within the gap to cancel against (i.e. the lower two panes in Figure 1.1 can be present within the gap but the upper two cannot). This is the origin of the "global" [1, 2] logarithmic correction (Table 1.1).

The global logarithms are a product of "primary" virtual in-gap gluons which have vertices only on the fermion lines (see Figure 1.2). Thus there are no gluons

Table 1.1: Global Logarithms

	rapidity inside gap	rapidity outside gap
$k_T > Q_0$	virtual gluons only	real/virtual cancellation
$k_T < Q_0$	real/virtual cancellation	real/virtual cancellation

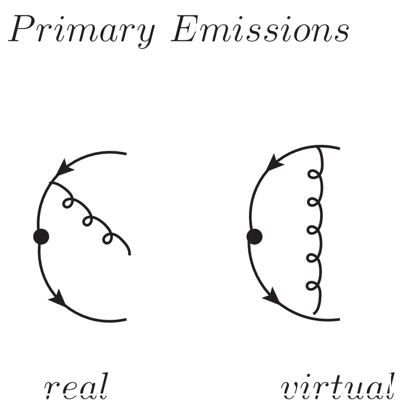

Figure 1.2: Primary Gluon Emissions

outside the gap contributing to the global correction.

There is, however, a further source of virtual corrections that need to be considered in the in-gap region; the so called "non-global" logarithms [4, 8]. Primary real emissions may be dressed with secondary virtual corrections connecting the existing quark, antiquark and gluon lines in all possible ways (see Figure 1.3, where the left hand pane is understood to encompass secondary virtual gluons connecting outgoing real partons in all possible ways). Primary eikonal virtual gluons may also be dressed with secondary virtual gluons (Figure 1.3 centre pane where the darker gluon line represents the eikonal gluon).

The secondary virtual corrections are cancelled, again as an application of

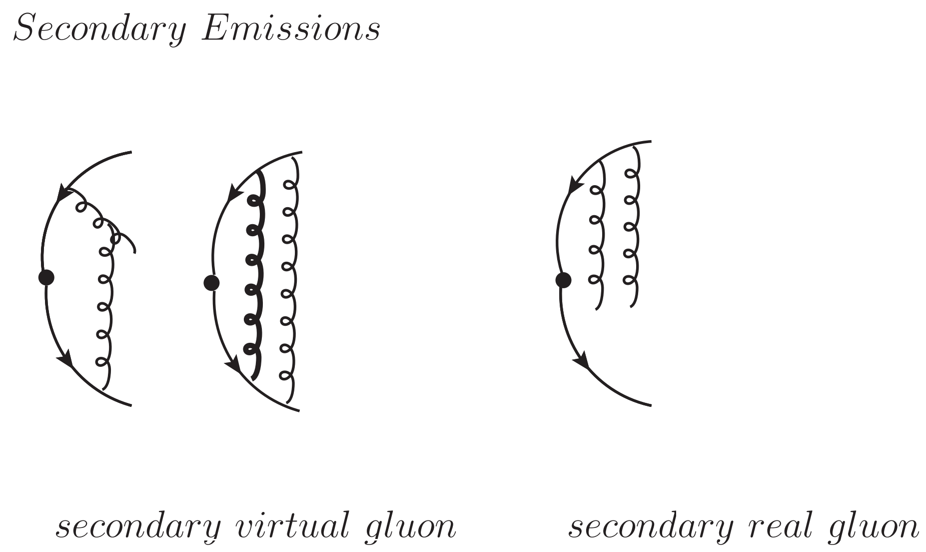

Figure 1.3: Secondary Gluon Emissions

BNT, by further real gluon radiation emitted off of quark-antiquark lines (after a primary emission) or off primary real gluon lines (Figure 1.3) where the right hand pane is understood to represent a second real gluon emitted from any of the three outgoing partons).

Again, both in the gap with transverse momenta below Q_0 and out of the gap at all momenta, both real and virtual secondary gluons can coexist, leading to complete cancellation of logarithms. Crucially however, the secondary virtual corrections may originate from primary real and (subsequent to) eikonal gluons lying outside the gap with transverse momenta above Q_0, and may radiate back into the gap. Thus in Figure 1.4 a cut across a two gluon diagram is illustrated. The right hand pane represents a primary out of gap real gluon with $k_T > Q_0$ radiating a secondary virtual gluon with a lower k_T but still above Q_0 back into

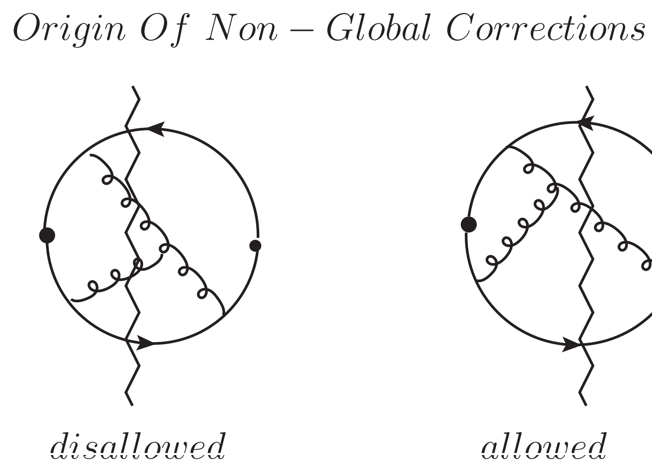

Figure 1.4: Origin Of Non-Global Corrections

the gap. The left hand pane shows the diagram that would be needed for Bloch-Nordsieck cancellation of this virtual gluon to take place. This real gluon with $k_T > Q_0$ cannot exist within the gap and so there is nothing for the virtual gluon to cancel against.

To account for all of the leading logarithms, it is therefore necessary to include both the global logarithms arising from no gluons outside of the gap and the non-global logarithms arising from an arbitary number of soft real and virtual (eikonal) emissions with $k_T > Q_0$ outside the gap, dressed with virtual gluons with $k_T > Q_0$ radiating back into the gap.

The colour algebra is reasonably straightforward when exponentiating virtual corrections as described above. However when calculating the out of gap real gluons responsible for the non-global correction, the colour algebra becomes very complex. One approach to calculations has therefore been to approximate the colour structure using the large (number of colours) N_c approximation. This has been done both numerically [4] and by the derivation of the Banfi-Marchesini-Smye (BMS) evolution equation which resums all the leading global and non-global logarithms [9].

It would be valuable to assess the accuracy of the BMS equation by comparison with a method that places no approximation on the colour structure but accounts for all the leading logarithms to a fixed order. This body of work attempts to calculate components of such a cross-section that can subsequently be used to compare with the BMS equation.

This thesis will describe the derivation of the eikonal rules from Feynman rules, illustrating where the momentum and colour factors come from. The colour operators will then be used to derive the "Anomalous Dimension Matrices" (ADM) $\mathbf{\Gamma}$ [5] that describe the addition of the virtual in-gap gluon dressing. These will be derived for zero, one and two gluons outside the gap.

The non-global corrections for the out of gap gluons involve the sum of the in-gap virtual dressing of real out of gap gluons (emitted by the $\mathbf{D}$ matrices) plus the dressing of the eikonal virtual out of gap gluons (emitted by the $\boldsymbol{\gamma}$ matrices) [5].

In addition to the ADM's then, the $\boldsymbol{D}$ and $\boldsymbol{\gamma}$ emission matrices will be derived for one and two gluons outside the gap. Finally the corrections will be used to calculate the modification to the Born cross-section to $O(\alpha_s)^3$ for zero and one gluon outside the gap and to $O(\alpha_s)^2$ for two gluons outside the gap. To this order, the ADM for two gluons outside the gap (which are of $O(\alpha_s)^3$ and higher) are not necessary; they will be used to examine the BMS equation up to $O(\alpha_s)^3$ in further work.

Chapter 2

Methods

2.1 Derivation Of Eikonal Rules

The principle behind the eikonal approximation is that the four momentum of the parent parton either before the emission of a soft gluon or after the absorption of a soft gluon can be approximated as the four momentum of the parent parton alone.

As an example of the derivation of the eikonal rules from the Feynman rules, the emission of a soft gluon off an outgoing antiquark from a generic quark-antiquark scattering process ($\mathbf{\Lambda}$, represented by the black circle in Figure 2.1) will be derived.

The amplitude has the form

$$\boldsymbol{M} = \bar{u}\left(p_4, s_4\right) \mathbf{\Lambda} \left(i \frac{\left(p_3 + k\right) \gamma_\mu + m}{\left(p_3 + k\right)^2 - m^2 + i\varepsilon} \right) \varepsilon^*_\sigma \left(k, \lambda\right) \left(-i g_s t_{a\,ji} \gamma^\sigma\right) v\left(p_3, s_3\right) \quad (2.1)$$

where m is the mass of the antiquark and g_s is the strong coupling constant; p_3, s_3

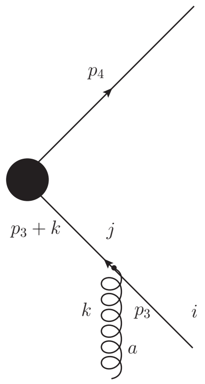

Figure 2.1: Eikonal Emission Of A Gluon From An Antiquark

and p_4, s_4 are the four momentum and spin of the antiquark and quark respectively; k and λ are the four momentum and polarization of the gluon respectively; j and i are the colour indices of the antiquark before and after emission of the gluon of colour a; $-t_{aji}$ are the components of the colour operator $-\boldsymbol{t}_a$ (a Gell-Mann matrix) for the emission of a gluon from the antiquark line.

For massless particles $m = 0, p^2 = 0$ and for soft gluons all $k^\mu <<$ maximum p^μ . Therefore $(p_3 + k)^2 \backsimeq 2p_3.k$ and $\nu(p_3 + k) \backsimeq \nu(p_3)$.

The antiquark propagator is:

$$i\frac{\left((p_3 + k)^\mu \gamma_\mu + m\right)}{(p_3 + k)^2 - m^2 + i\varepsilon} \tag{2.2}$$

and

$$(p_3+k)^{\mu}\gamma_{\mu}+m=\sum_{s_3',s_3} v\left(p_3+k,s_3'\right)\overline{v}\left(p_3+k,s_3\right)$$

therefore

$$\boldsymbol{M} \;=\; \overline{u}\left(p_4,s_4\right)\boldsymbol{\Lambda}\left(i\frac{\sum\limits_{s_3',s_3} v\left(p_3,s_3'\right)\overline{v}\left(p_3,s_3\right)}{2p_3.k+i\varepsilon}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)\left(-ig_s t_{aji}\gamma^{\sigma}\right)v\left(p_3,s_3\right)$$

If the amplitude for the process without the radiated soft gluon is $\boldsymbol{M}_0$, where

$$\boldsymbol{M}_0=\sum_{s_4,s_3'}\overline{u}\left(p_4,s_4\right)\boldsymbol{\Lambda}v\left(p_3,s_3'\right) \tag{2.3}$$

then

$$\boldsymbol{M}=\boldsymbol{M}_0\left(i\frac{\overline{v}\left(p_3s_3'\right)}{2p_3.k+i\varepsilon}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)\left(-ig_s t_{aji}\gamma^{\sigma}\right)v\left(p_3,s_3\right). \tag{2.4}$$

As

$$-\gamma_{\mu}p^{\mu}v\left(p,s\right)=mv\left(p,s\right) \tag{2.5}$$

by Dirac, then

$$-\overline{v}(p,s)\gamma_\mu v(p,s)p^\mu \quad = \quad m\overline{v}(p,s)v(p,s)$$

so

$$-\overline{v}(p,s)\gamma_\mu v(p,s)p^\mu = 2m^2. \tag{2.6}$$

Now

$$\overline{v}(p,s)\gamma_\mu v(p,s) = Ap_\mu \tag{2.7}$$

where A is a constant, therefore

$$-Ap^2 = 2m^2 \tag{2.8}$$

so

$$A = -2 \tag{2.9}$$

therefore

$$\overline{v}(p_3,s_3)\gamma^\sigma v\left(p_3,s_3'\right) = -2p_3^\sigma \delta_{s_3 s_3'} \tag{2.10}$$

so

$$\boldsymbol{M} = \boldsymbol{M}_0 \frac{i}{(2p_3.k + i\varepsilon)}\left(-t_{aji}\right)\left(-2ig_s p_3^\sigma \varepsilon_\sigma^*(k,\lambda)\right). \tag{2.11}$$

The eikonal propagator is thus:

$$\frac{i}{(2p_3.k + i\varepsilon)} \tag{2.12}$$

Whilst the eikonal vertex is:

$$\left(-t_{aji}\right)\left(-2ig_s p_3^{\sigma}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right) \tag{2.13}$$

The derivation of the eikonal propagators and vertices for (incoming and outgoing) quarks and gluons follows the same method (see Table 2.1). The four momentum terms $\frac{p_3^{\sigma}}{2p_3.k}$ and colour terms $-t_{aji}$ may then be used to calculate the required emission and dressing matrices.

2.2 Eikonal Rules.

Table 2.1: Eikonal Rules For Radiated Gluons

Process	Propagator	Vertex
Outgoing Quark	$\frac{i}{(2p.k+i\varepsilon)}$	$\left(t_{aij}\right)\left(-2ig_s p^{\sigma}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)$
Outgoing Anti-quark	$\frac{i}{(2p.k+i\varepsilon)}$	$\left(-t_{aji}\right)\left(-2ig_s p^{\sigma}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)$
Incoming Quark	$\frac{i}{(2p.k-i\varepsilon)}$	$\left(-t_{aji}\right)\left(-2ig_s p^{\sigma}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)$
Incoming Anti-quark	$\frac{i}{(2p.k-i\varepsilon)}$	$\left(t_{aij}\right)\left(-2ig_s p^{\sigma}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)$
Incoming Gluon	$\frac{i}{(2p.k-i\varepsilon)}$	$\left(if_{acb}\right)\left(-2ig_s p^{\sigma}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)$
Outgoing Gluon	$\frac{i}{(2p.k+i\varepsilon)}$	$\left(-if_{abc}\right)\left(-2ig_s p^{\sigma}\right)\varepsilon_{\sigma}^{*}\left(k,\lambda\right)$

The convention of the first index (a) referring to the colour of the emitted eikonal gluon has been adopted for the colour operators. The order of the second and third indices are read against fermion flow for fermions and against the four momentum of the parent line for gluons. This is illustrated in Figure 2.2 (with reference to the colour operators in Table 2.1) for gluons emitted from outgoing

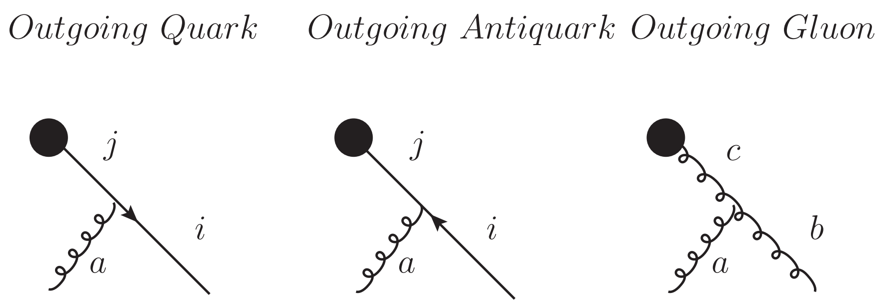

Figure 2.2: Colour Index Labelling Of Eikonal Emissions

quark, antiquark and gluon lines. The arrow on the fermion line indicates fermion flow.

All colour operators are expressed in covariant tensor form for clarity. Early alphabetic labels (a,b,c,e,g,h) are used for the colour of gluon lines whilst mid-alphabetic labels (i,j,k,l,m,n) are reserved for quark/antiquark lines.

The colour factor for an incoming (with the indices read against fermion flow) quark or outgoing antiquark line is $-t_{aji}$, whist for an incoming antiquark or outgoing quark the colour factor is t_{aij}. For an incoming gluon radiating a gluon the colour factor is if_{acb}, whilst for an outgoing gluon radiating a gluon the colour factor is $-if_{abc}$ (with the indices read against the four momentum flow of the parent line) [10].

The following calculations were all performed with N = 3. In further work it is intended that the results will be presented in the full N_c form.

2.3 Colour Algebra

The following identities are extensively used in the succeeding calculations:

$$
\begin{aligned}
\left(t_{aij}\right)^* &= t_{aji} \\
t_{aii} &= 0 \\
t_{aij}t_{ajk} &= \frac{4}{3}\delta_{ik} \\
t_{aij}t_{bji} &= \frac{\delta_{ab}}{2} \\
t_{aij}t_{bjk}t_{akl} &= -\frac{1}{6}t_{bil} \\
t_{aij}t_{bjk}t_{cki} &= \frac{1}{4}\left(if_{abc}+d_{abc}\right) \\
t_{aij}t_{bjk}\left(-if_{abc}\right) &= \frac{3}{2}t_{cik} \\
t_{aij}t_{bjk}\left(d_{abc}\right) &= \frac{5}{6}t_{cik} \\
d_{abc} &= d_{acb} \\
d_{abb} &= 0 \\
d_{ace}d_{bce} &= \frac{5}{3}\delta_{ab} \\
if_{abc} &= -if_{acb} \\
\left(if_{abc}\right)^* &= -if_{abc} = if_{acb} \\
d_{cab}f_{cab} &= 0 \\
f_{ace}f_{bce} &= 3\delta_{ab} \\
if_{bag}if_{gba}if_{acb} &= \frac{3}{2}if_{abc} \\
if_{bag}if_{gba}d_{acb} &= \frac{3}{2}d_{abc} \\
if_{bag}d_{gba}d_{acb} &= \frac{5}{6}if_{abc}
\end{aligned}
$$

2.4 Calculation Of Anomalous Dimension Matrices

The colour "basis" for a scattering process represents the independent colour structures, expressed as tensors in colour space, that contribute to the process [10]. The ADM's Γ act to dress the amplitude on which they act with one virtual gluon in all possible ways [5]. Their components are found by forming the inner products of the ADM with the colour basis tensors for the amplitude in question. For example, for a scattering process that has two basis tensors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, the components of the ADM are found by evaluating:

$$\begin{bmatrix} <\boldsymbol{c}_1|\boldsymbol{\Gamma}|\boldsymbol{c}_1> & <\boldsymbol{c}_1|\boldsymbol{\Gamma}|\boldsymbol{c}_2> \\ <\boldsymbol{c}_2|\boldsymbol{\Gamma}|\boldsymbol{c}_1> & <\boldsymbol{c}_2|\boldsymbol{\Gamma}|\boldsymbol{c}_2> \end{bmatrix} \tag{2.14}$$

The ADM for an azimuthally symmetric rapidity gap of length Y in a colour basis independent notation for $2 \to n$ particle scattering is [10]:

$$\begin{aligned} \boldsymbol{\Gamma} &= \frac{1}{2}Y\boldsymbol{T}_t^2 + i\pi\boldsymbol{T}_1.\boldsymbol{T}_2 + \frac{1}{4}\sum_{i\in F}\rho(Y;2|y_i|)\boldsymbol{T}_i^2 \\ &+ \frac{1}{2}\sum_{(i<j)\in L}\lambda\left(Y;|y_i|+|y_j|,|\phi_i-\phi_j|\right)\boldsymbol{T}_i.\boldsymbol{T}_j \\ &+ \frac{1}{2}\sum_{(i<j)\in R}\lambda\left(Y;|y_i|+|y_j|,|\phi_i-\phi_j|\right)\boldsymbol{T}_i.\boldsymbol{T}_j \end{aligned} \tag{2.15}$$

$\boldsymbol{T}_i$ is the generic colour operator of parton i, whilst the notation is modified slightly to $\boldsymbol{T}_{ai}$ for the generic operator that emits a gluon of colour a from parton i. $\boldsymbol{T}_t^2 = (\boldsymbol{T}_1+\boldsymbol{T}_3+\boldsymbol{T}_k)^2 = (\boldsymbol{T}_2+\boldsymbol{T}_4)^2$ is the colour transferred in the t channel. $\boldsymbol{T}_1+$

$\boldsymbol{T}_2$ are the colour operators of the incoming partons, whilst $\boldsymbol{T}_3$ and $\boldsymbol{T}_4$ are the operators for the outgoing fermions and $\boldsymbol{T}_k$ are the operators for the out of gap gluons.

The rapidity gap between the principal jets is Y; the rapidity of jet i is y_i (whether this is a fermion or out of gap gluon) and the jet cone radius is R. Thus if Δy is the magnitude of the difference in jet rapidities then $\Delta y = |y_3| + |y_4| = Y + 2R$.

ρ is a jet function where:

$$\rho(Y;2|y|) \quad = \quad \ln \frac{\sinh(|y|/2 + Y/2)}{\sinh(|y|/2 - Y/2)} - Y$$

λ is another jet function where:

$$\lambda(Y;|y_i| + |y_j|, |\phi| \quad = \quad \frac{1}{2} \ln \frac{\cosh\left(|y_i| + \left|y_j\right| + Y\right) - sgn\left(y_{i,j}\right)\cos(\phi)}{\cosh\left(|y_i| + \left|y_j\right| - Y\right) - sgn\left(y_{i,j}\right)\cos(\phi)} - Y$$

F are the final state partons and L and R refer to (both initial and final state) partons on the left and right hand sides of the gap. The $i < j$ device is used to ensure that each parton is counted only once [6, 10].

In general the ADM 's for the dressing of gluons on different sides of the gap will not be the same due to the different combination of colour operators present in the Y and λ terms. The method of calculation is clearly the same whichever side the out of gap gluons are emitted on. For consistency all of the ADM calculations for $e^- e^+ \rightarrow q\bar{q}$ will be for gluons emitted on the left.

By way of example in order to check the above eikonal rules and demonstrate the method of calculating the ADM components, an element (the term proportional to Y) of the $\mathbf{\Gamma}_{13}$ matrix component for quark-gluon scattering as calculated in [10] is reproduced. The ADM and basis vector $\boldsymbol{c}_3$ is are taken from this paper. The 13 subscript on the ADM indicates that the basis is a t channel basis.

2.4.1 A Cross-Check

In this case:

$$\begin{aligned}\mathbf{\Gamma}_{13} &= \frac{1}{2}Y\boldsymbol{T}_t^2 + i\pi\boldsymbol{T}_1.\boldsymbol{T}_2 + \frac{1}{4}\rho_{jet}\left(Y,|\Delta Y|\right)\left(\boldsymbol{T}_3^2+\boldsymbol{T}_4^2\right)\\ \boldsymbol{c}_3 &= \frac{1}{\sqrt{12}}if_{cbg}t_{ckn}\end{aligned}$$

Identifying the incoming and outgoing quarks as partons 1 and 3 and the incoming and outgoing gluons as partons 2 and 4, means $\boldsymbol{T}_1=\boldsymbol{t}_1$ and $\boldsymbol{T}_3=\boldsymbol{t}_3$. Thus if $\boldsymbol{t}_t$ is the colour exchanged in the t channel, then:

$$\boldsymbol{t}_t = (\boldsymbol{t}_1+\boldsymbol{t}_3)$$

therefore

$$\boldsymbol{t}_t^2 = \boldsymbol{t}_1^2+\boldsymbol{t}_3^2+2\boldsymbol{t}_1.\boldsymbol{t}_3 \tag{2.16}$$

To calculate the $\frac{1}{2}Y\boldsymbol{T}_t^2$ element $< \boldsymbol{c}_3\left|\frac{1}{2}Y\left(\boldsymbol{t}_1^2+\boldsymbol{t}_3^2+2\boldsymbol{t}_1.\boldsymbol{t}_3\right)\right|\boldsymbol{c}_3 >$ must be evaluated. The three virtual gluon elements are illustrated (both the incoming and

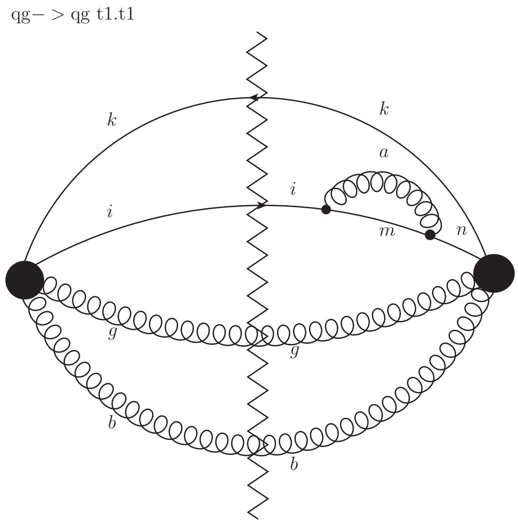

Figure 2.3: Cross-Check (First Component)

outgoing partons are represented. The large black circle represents the scattering process whilst the small circles represent the gluon vertices) and calculated below. The first combination is the $\mathbf{t}_1.\boldsymbol{t}_1$ gluon (see Figure 2.3) where:

$$
\begin{aligned}
\left(\frac{Y}{2}\right)\frac{1}{\sqrt{12}}\left(if_{ebg}t_{eki}\right)^{*}\left(-t_{ami}\right)\left(-t_{anm}\right)\frac{1}{\sqrt{12}}\left(if_{cbg}t_{ckn}\right) &= \frac{Y}{24}t_{eik}if_{egb}\left(\frac{4}{3}\delta_{ni}\right)if_{cbg}t_{ckn} \\
&= -\frac{Y}{18}t_{enk}t_{ckn}f_{egb}f_{cbg} \\
&= -\frac{Y}{18}\left(-3\delta_{ce}\right)t_{enk}t_{ckn} \\
&= \frac{Y}{6}t_{cnk}t_{ckn} \\
&= \frac{Y}{6}\left(\frac{\delta_{cc}}{2}\right) \\
&= \frac{2Y}{3}
\end{aligned}
$$

qg− > qg t3.t3

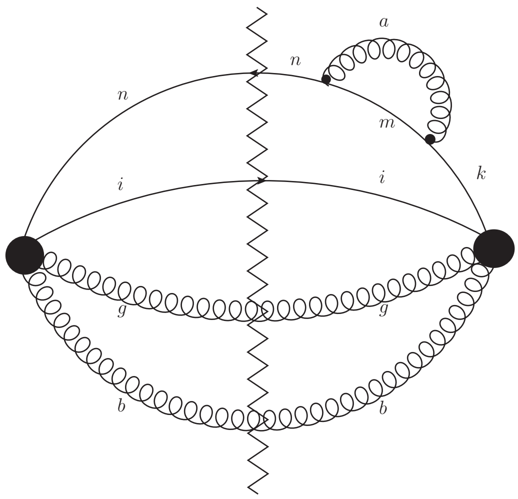

Figure 2.4: Cross-Check (Second Component)

The second combination is the $\boldsymbol{t}_3.\boldsymbol{t}_3$ gluon (see Figure 2.4) where

$$\begin{aligned}
\left(\frac{Y}{2}\right)\frac{1}{\sqrt{12}}\left(if_{ebg}t_{eni}\right)^{*}t_{anm}t_{amk}\frac{1}{\sqrt{12}}\left(if_{cbg}t_{cki}\right) &= \frac{Y}{24}t_{ein}if_{egb}\left(\frac{4}{3}\delta_{nk}\right)if_{cbg}t_{cki} \\
&= -\frac{Y}{18}t_{eik}t_{cki}f_{egb}f_{cbg} \\
&= -\frac{Y}{18}\left(-3\delta_{ce}\right)t_{eik}t_{cki} \\
&= \frac{Y}{6}t_{cik}t_{cki} \\
&= \frac{Y}{6}\left(\frac{\delta_{cc}}{2}\right) \\
&= \frac{2Y}{3}
\end{aligned}$$

The third combination is the $\boldsymbol{t}_1.\boldsymbol{t}_3$ gluon (see Figure 2.5) where

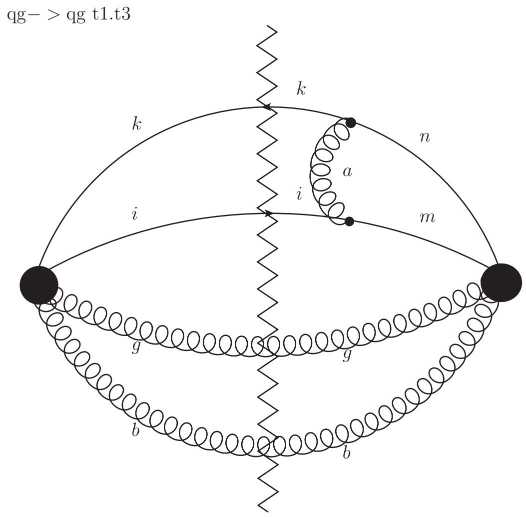

Figure 2.5: Cross-Check (Third Component)

$$
\begin{aligned}
\left(\frac{Y}{2}\right)\frac{1}{\sqrt{12}}\left(if_{ebg}t_{eki}\right)^{*}t_{akn}\left(-t_{ami}\right)\frac{1}{\sqrt{12}}\left(if_{cbg}t_{cnm}\right) &= -\frac{Y}{24}\left(if_{egb}t_{eik}\right)t_{akn}t_{ami}\left(if_{cbg}t_{cnm}\right) \\
&= \frac{Y}{24}\left(-3\delta_{ec}\right)t_{akn}t_{ami}t_{eik}t_{cnm} \\
&= -\frac{Y}{8}t_{cik}\left(t_{akn}t_{cnm}t_{ami}\right) \\
&= -\frac{Y}{8}t_{cik}\left(-\frac{1}{6}t_{cki}\right) \\
&= \frac{Y}{48}t_{cik}t_{cki} \\
&= \frac{Y}{48}\left(\frac{\delta_{cc}}{2}\right) \\
&= \frac{Y}{12}
\end{aligned}
$$

Therefore $2\boldsymbol{t}_1.\boldsymbol{t}_3 = \frac{Y}{6}$ and thus the sum of the three gluon contributions is

$\frac{2Y}{3}+\frac{2Y}{3}+\frac{Y}{6}=\frac{3Y}{2}$ in agreement with the same matrix element in [10].

Chapter 3

Results

3.1 ADM Calculations For $e^-e^+ \to q\bar{q}$

3.1.1 Zero Gluons Outside The Gap

For the process $e^-e^+ \to q\bar{q}$ the sum of the quark and antiquark colours, by colour conservation, must be zero, i.e. the un-normalized tensor must be δ_{kl} where k and l are the colour labels of the quark and antiquark respectively. As $\delta_{kl}\delta_{kl} = \delta_{kk} = 3$, then the normalization factor must be $\frac{1}{\sqrt{3}}$ and thus the normalized basis tensor for this process is $\frac{1}{\sqrt{3}}\delta_{kl}$.

With reference to the general ADM, $\boldsymbol{T}_1$+ $\boldsymbol{T}_2$ are the generic colour operators of incoming partons (both being zero for $e^-e^+ \to q\bar{q}$) whilst $\boldsymbol{T}_3$ and $\boldsymbol{T}_4$ are the operators for the outgoing fermions. Identifying $\boldsymbol{T}_3 = -\boldsymbol{t}_3$ as the colour operator for the anti-quark on the left hand side of the gap, $\boldsymbol{T}_4 = \boldsymbol{t}_4$ for the quark on the right respectively and $\boldsymbol{T}_k = -i\boldsymbol{f}_k$ the colour operator for outgoing gluons for one and two gluons outside the gap. $\boldsymbol{T}_t$ the colour transferred in the t channel, is therefore $\boldsymbol{T}_t^2$= $(\boldsymbol{T}_3)^2$= $(\boldsymbol{T}_4)^2$. Thus with only two coloured outgoing particles and values of zero for the λ functions (as there is only one outgoing particle on each side of the gap), the ADM for zero gluons outside the gap $\mathbf{\Gamma}_0$ simplifies to:

$$\mathbf{\Gamma}_0 = \frac{1}{2}Y\boldsymbol{T}_3^2 + \frac{1}{4}\rho(Y, 2|y_3|)\boldsymbol{T}_3^2 + \frac{1}{4}\rho(Y, 2|y_4|)\boldsymbol{T}_4^2$$

which as $\boldsymbol{T}_3^2 = \boldsymbol{T}_4^2$ further simplifies to:

$$\mathbf{\Gamma}_0 = \frac{1}{2}Y\boldsymbol{T}_3^2 + \frac{1}{4}\rho(Y, 2|y_3|)\boldsymbol{T}_3^2 + \frac{1}{4}\rho(Y, 2|y_4|)\boldsymbol{T}_3^2$$

Thus the $\boldsymbol{T}_3^2$ element is (see Figure 3.1):

$$\begin{aligned}\left(\frac{1}{\sqrt{3}}\delta_{kn}\right)^*(-t_{hlm})(-t_{hmn})\left(\frac{1}{\sqrt{3}}\delta_{kl}\right) &= \frac{1}{3}\delta_{nk}t_{hlm}t_{hmn}\delta_{kl} \\ &= \frac{1}{3}t_{hlm}t_{hml} \\ &= \frac{4}{3}\end{aligned}$$

Therefore

$$\mathbf{\Gamma}_0 = \frac{2}{3}Y + \frac{1}{3}\rho(Y, 2|y_3|) + \frac{1}{3}\rho(Y, 2|y_4|)$$

Zero Gluons Outside The Gap t3.t3

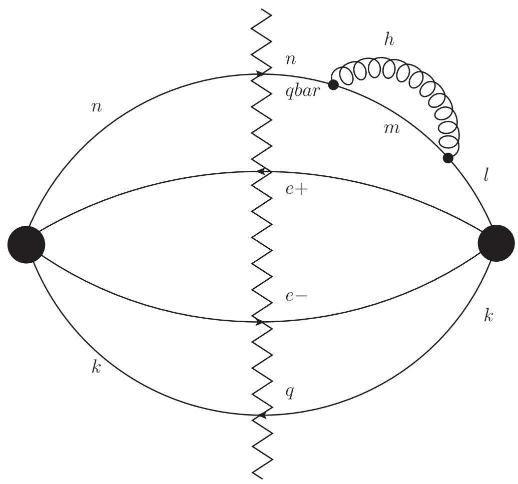

Figure 3.1: Zero Gluons Outside The Gap Dressing

3.1.2 One Gluon Outside The Gap

To calculate the virtual dressing for one gluon outside of the gap a basis tensor with one quark, one antiquark and one gluon index is needed; the basis tensor (there is clearly only one) for this process is therefore t_{akl}. As $t_{akl}t_{alk} = 4$ then the normalization coefficient is $\frac{1}{2}$ and the normalized basis tensor is $\frac{1}{2}t_{akl}$. The ADM $\mathbf{\Gamma}_1$ is:

$$\mathbf{\Gamma}_1 = \frac{1}{2}Y\boldsymbol{T}_t^2 + \frac{1}{4}\sum_{i\in F}\rho(Y;2|y_i|)\boldsymbol{T}_i^2 + \frac{1}{2}\sum_{(i<j)\in L}\lambda\left(Y;|y_i|+|y_j|,|\phi_i-\phi_j|\right)\boldsymbol{T}_i.\boldsymbol{T}_j$$

for one gluon emitted to the left, the anti-quark side of the gap, identifying $\bar{q}$ and q as partons 3 and 4 respectively and the gluon as parton 5. Therefore $\boldsymbol{T}_t^2$= ($\boldsymbol{T}_3$+

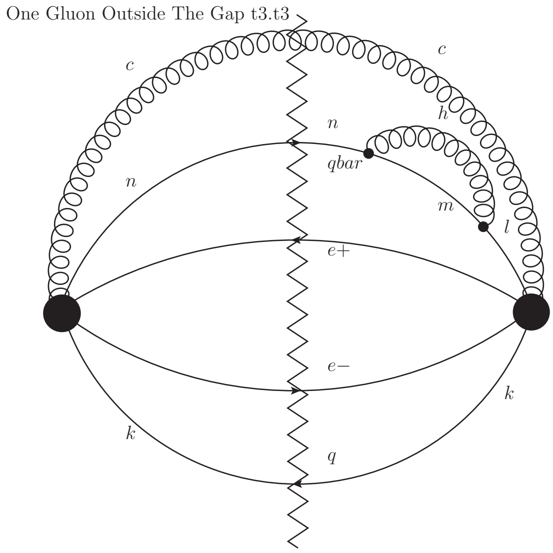

Figure 3.2: One Gluon Outside The Gap Dressing

$\boldsymbol{T}_5)^2$ and the ADM transforms to:

$$\begin{aligned}\boldsymbol{\Gamma}_1 &= \frac{1}{2}Y(\boldsymbol{T}_3+\boldsymbol{T}_5)^2+\frac{1}{4}\rho(Y;2|y_3|)\boldsymbol{T}_3^2+\frac{1}{4}\rho(Y;2|y_4|)\boldsymbol{T}_4^2\\ &+ \frac{1}{4}\rho(Y;2|y_5|)\boldsymbol{T}_5^2+\frac{1}{2}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)\boldsymbol{T}_3.\boldsymbol{T}_5\end{aligned}$$

Thus the colour operator combinations we need are: $\boldsymbol{T}_3^2, \boldsymbol{T}_5^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_5.\boldsymbol{T}_3$ and $\boldsymbol{T}_4^2$. As $\boldsymbol{T}_3.\boldsymbol{T}_5 = \boldsymbol{T}_5.\boldsymbol{T}_3$ and $\boldsymbol{T}_3^2 = \boldsymbol{T}_4^2$ this simplifies to $\boldsymbol{T}_3^2, \boldsymbol{T}_5^2$ and $\boldsymbol{T}_3.\boldsymbol{T}_5$.

The $\boldsymbol{T}_3$ operator is $-\boldsymbol{t}_3$. Therefore the $\boldsymbol{T}_3^2$ element is (see Figure 3.2):

$$
\begin{aligned}
\frac{1}{2}(t_{ckn})^*(-t_{hlm})(-t_{hmn})\frac{1}{2}(t_{ckl}) &= \frac{1}{4}t_{cnk}t_{hlm}t_{hmn}t_{ckl} \\
&= \frac{1}{4}\left(\frac{4}{3}\delta_{ln}\right)t_{cnk}t_{ckl} \\
&= \frac{1}{3}t_{clk}t_{ckl} \\
&= \frac{1}{3}\left(\frac{\delta_{cc}}{2}\right) \\
&= \frac{4}{3}
\end{aligned}
$$

The $\boldsymbol{T}_5$ operator is $-i\boldsymbol{f}_a$ thus the $\boldsymbol{T}_5^2$ element is:

$$
\begin{aligned}
\frac{1}{2}(t_{akl})^*(-if_{hbc})(-if_{hab})\frac{1}{2}(t_{ckl}) &= \frac{1}{4}t_{alk}f_{chb}f_{ahb}t_{ckl} \\
&= \frac{1}{4}(3\delta_{ca})t_{alk}t_{ckl} \\
&= \frac{3}{4}t_{clk}t_{ckl} \\
&= \frac{3}{4}\left(\frac{\delta_{cc}}{2}\right) \\
&= 3
\end{aligned}
$$

The $\boldsymbol{T}_3.\boldsymbol{T}_5$ element is thus:

$$\begin{aligned}
\frac{1}{2}(t_{ekm})^*(-t_{hlm})(-if_{hec})\frac{1}{2}(t_{ckl}) &= \frac{1}{4}t_{emk}t_{hlm}(-if_{hce})t_{ckl} \\
&= \frac{1}{4}t_{emk}t_{ckl}t_{hlm}if_{che} \\
&= \frac{1}{4}t_{emk}\left(-\frac{3}{2}t_{ekm}\right) \\
\text{as } t_{ckl}t_{hlm}if_{che} = -\frac{3}{2}t_{ekm} & \\
&= -\frac{3}{8}t_{emk}t_{ekm} \\
&= -\frac{3}{8}\left(\frac{\delta_{ee}}{2}\right) \\
&= -\frac{3}{2}
\end{aligned}$$

Thus

$$\begin{aligned}
\boldsymbol{\Gamma}_1 &= \frac{1}{2}Y(\frac{4}{3}+2\left(-\frac{3}{2}\right)+3)+\frac{1}{4}\rho(Y;2|y_3|)(\frac{4}{3})+\frac{1}{4}\rho(Y;2|y_4|)(\frac{4}{3}) \\
&+ \frac{1}{4}\rho(Y;2|y_5|)(3)+\frac{1}{2}\lambda(Y;|y_3|+|y_5|,|\phi_3-\phi_5|)(-\frac{3}{2}) \\
&= \frac{2}{3}Y+\frac{1}{3}\rho(Y;2|y_3|)+\frac{1}{3}\rho(Y;2|y_4|)+\frac{3}{4}\rho(Y;2|y_5|) \\
&- \frac{3}{4}\lambda(Y;|y_3|+|y_5|,|\phi_3-\phi_5|)
\end{aligned}$$

3.1.3 Two Gluons Outside The Gap

For two gluons outside the gap a basis is needed that connects a quark, anti-quark and two gluons. The basis chosen for this calculation is the $gg \to q\overline{q}$ basis from [10] where:

$$\boldsymbol{c}_1 = \frac{1}{\sqrt{24}}\delta_{ab}\delta_{kl}$$

$$\boldsymbol{c}_2 = \sqrt{\frac{3}{20}}d_{cab}t_{ckl}$$

$$\boldsymbol{c}_3 = \frac{1}{\sqrt{12}}if_{cab}t_{ckl}$$

The convention of putting the internal gluon line first (c) has been adopted for the basis vectors. The ADM will again be different for the cases of two gluons on the right and one on each side. The method of calculation is the same in each case and so the ADM for two gluons emitted on the left (the antiquark side of the gap) will be calculated.

The ADM for two gluons on the left $\mathbf{\Gamma}_2$ is:

$$\mathbf{\Gamma}_2 = \frac{1}{2}Y\boldsymbol{T}_t^2 + \frac{1}{4}\sum_{i\in F}\rho(Y;2|y_i|)\boldsymbol{T}_i^2 + \frac{1}{2}\sum_{(i<j)\in L}\lambda\left(Y;|y_i|+\left|y_j\right|,\left|\phi_i-\phi_j\right|\right)\boldsymbol{T}_i.\boldsymbol{T}_j \tag{3.1}$$

Therefore identifying $\bar{q}$ and q as partons 3 and 4 and the gluons as partons 5 and 6 respectively, (with $\boldsymbol{T}_t = \boldsymbol{T}_3 + \boldsymbol{T}_5 + \boldsymbol{T}_6$), transforms the ADM to:

$$
\begin{aligned}
\mathbf{\Gamma}_2 &= \frac{1}{2}Y(\boldsymbol{T}_3+\boldsymbol{T}_5+\boldsymbol{T}_6)^2+\frac{1}{4}\rho(Y;2|y_3|)\boldsymbol{T}_3^2+\frac{1}{4}\rho(Y;2|y_4|)\boldsymbol{T}_4^2+\frac{1}{4}\rho(Y;2|y_5|)\boldsymbol{T}_5^2 \\
&+ \frac{1}{4}\rho(Y;2|y_6|)\boldsymbol{T}_6^2+\frac{1}{2}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)\boldsymbol{T}_3.\boldsymbol{T}_5 \\
&+ \frac{1}{2}\lambda\left(Y;|y_3|+|y_6|,|\phi_3-\phi_6|\right)\boldsymbol{T}_3.\boldsymbol{T}_6+\frac{1}{2}\lambda\left(Y;|y_5|+|y_6|,|\phi_5-\phi_6|\right)\boldsymbol{T}_5.\boldsymbol{T}_6
\end{aligned}
$$

Thus the colour operator combinations required are:

$\boldsymbol{T}_3^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_3.\boldsymbol{T}_6, \boldsymbol{T}_5^2, \boldsymbol{T}_5.\boldsymbol{T}_6$ (as $\boldsymbol{T}_3^2=\boldsymbol{T}_4^2$ and $\boldsymbol{T}_5^2=\boldsymbol{T}_6^2$).

Therefore:

$$
\begin{aligned}
\mathbf{\Gamma}_2 &= \frac{1}{2}Y(\boldsymbol{T}_3^2+2\boldsymbol{T}_5^2+2\boldsymbol{T}_3.\boldsymbol{T}_5+2\boldsymbol{T}_3.\boldsymbol{T}_6+2\boldsymbol{T}_5.\boldsymbol{T}_6) \\
&+ \frac{1}{4}\rho(Y;2|y_3|)\boldsymbol{T}_3^2+\frac{1}{4}\rho(Y;2|y_4|)\boldsymbol{T}_3^2 \\
&+ \frac{1}{4}\rho(Y;2|y_5|)\boldsymbol{T}_5^2+\frac{1}{4}\rho(Y;2|y_6|)\boldsymbol{T}_5^2 \\
&+ \frac{1}{2}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)\boldsymbol{T}_3.\boldsymbol{T}_5+\frac{1}{2}\lambda\left(Y;|y_3|+|y_6|,|\phi_3-\phi_6|\right)\boldsymbol{T}_3.\boldsymbol{T}_6 \\
&+ \frac{1}{2}\lambda\left(Y;|y_5|+|y_6|,|\phi_5-\phi_6|\right)\boldsymbol{T}_5.\boldsymbol{T}_6
\end{aligned}
$$

Each of these combinations needs to be evaluated for each combination of basis tensors i.e. $\mathbf{\Gamma}_2$ for two gluons out of the gap is a 3 by 3 matrix.

$$
\begin{bmatrix}
\langle \boldsymbol{c}_1 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_1\rangle & \langle \boldsymbol{c}_1 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_2\rangle & \langle \boldsymbol{c}_1 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_3\rangle \\
\langle \boldsymbol{c}_2 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_1\rangle & \langle \boldsymbol{c}_2 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_2\rangle & \langle \boldsymbol{c}_2 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_3\rangle \\
\langle \boldsymbol{c}_3 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_1\rangle & \langle \boldsymbol{c}_3 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_2\rangle & \langle \boldsymbol{c}_3 \mid \mathbf{\Gamma}_2 \mid \boldsymbol{c}_3\rangle
\end{bmatrix}
$$

Thus for each matrix element we require the components

$\left\langle \boldsymbol{c}_i \mid \boldsymbol{T}_3^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_3.\boldsymbol{T}_6, \boldsymbol{T}_5^2, \boldsymbol{T}_5.\boldsymbol{T}_6 \mid \boldsymbol{c}_j \right\rangle$.

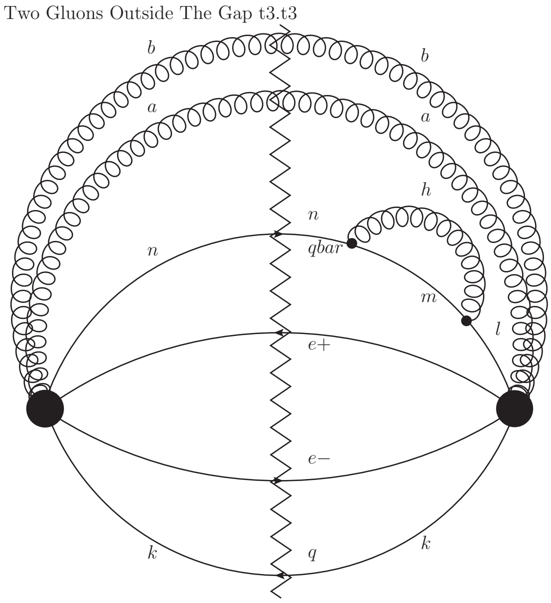

Figure 3.3: Two Gluon Outside The Gap Dressing

For $\langle \boldsymbol{c}_1 \mid \boldsymbol{\Gamma}_2 \mid \boldsymbol{c}_1 \rangle$ the following inner products are needed:
$\left\langle \boldsymbol{c}_1 \mid \boldsymbol{T}_3^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_3.\boldsymbol{T}_6, \boldsymbol{T}_5^2, \boldsymbol{T}_5.\boldsymbol{T}_6 \mid \boldsymbol{c}_1 \right\rangle$. The $\boldsymbol{T}_3$ operator is $-\boldsymbol{t}_h$ therefore the $\boldsymbol{T}_3^2$ element is (see Figure 3.3):

$$\begin{aligned}
\frac{1}{\sqrt{24}} (\delta_{ab}\delta_{kn})^* (-t_{hlm})(-t_{hmn}) \frac{1}{\sqrt{24}} (\delta_{ab}\delta_{kl}) &= \frac{1}{24} (\delta_{bb}) (\delta_{nl})\, t_{hlm} t_{hmn} \\
&= \frac{1}{24} .8 t_{hlm} t_{hml} \\
&= \frac{4}{3}
\end{aligned}$$

The $\boldsymbol{T}_5$ operator is $-i\boldsymbol{f}_a$ thus the $\boldsymbol{T}_5^2$ element is:

$$\begin{aligned}
\frac{1}{\sqrt{24}}\left(\delta_{eb}\delta_{kl}\right)^{*}\left(-if_{hga}\right)\left(-if_{heg}\right)\frac{1}{\sqrt{24}}\left(\delta_{ab}\delta_{kl}\right) &= -\frac{1}{24}\left(\delta_{kk}\right)f_{hgb}f_{hbg} \\
&= \frac{9}{24}\delta_{hh} \\
&= 3
\end{aligned}$$

The $\boldsymbol{T}_3.\boldsymbol{T}_5$ element is thus:

$$\frac{1}{\sqrt{24}}\left(\delta_{gb}\delta_{km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{hga}\right)\frac{1}{\sqrt{24}}\left(\delta_{ab}\delta_{kl}\right) = 0$$

as $\delta_{km}\delta_{kl}t_{hlm} = t_{hmm} = 0$

The $\boldsymbol{T}_3.\boldsymbol{T}_6$ element is:

$$\frac{1}{\sqrt{24}}\left(\delta_{ag}\delta_{km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{hgb}\right)\frac{1}{\sqrt{24}}\left(\delta_{ab}\delta_{kl}\right) = 0$$

as $\delta_{km}\delta_{kl}t_{hlm} = t_{hmm} = 0$

The $\boldsymbol{T}_5.\boldsymbol{T}_6$ element is:

$$\begin{aligned}
\frac{1}{\sqrt{24}}\left(\delta_{eg}\delta_{kl}\right)^{*}\left(-if_{hea}\right)\left(-if_{hgb}\right)\frac{1}{\sqrt{24}}\left(\delta_{ab}\delta_{kl}\right) &= -\frac{1}{24}\left(\delta_{kk}\right)f_{hgb}f_{hgb} \\
&= -\frac{3}{24}\left(3\delta_{hh}\right) \\
&= -3
\end{aligned}$$

Thus:

$$
\begin{aligned}
\boldsymbol{\Gamma}_{211} &= \frac{1}{2}Y(\frac{4}{3}+2.3+2.0+2.0-2.3)\\
&+ \frac{1}{4}\rho(Y;2|y_3|)\left(\frac{4}{3}\right)+\frac{1}{4}\rho(Y;2|y_4|)\left(\frac{4}{3}\right)\\
&+ \frac{1}{4}\rho(Y;2|y_5|)(3)+\frac{1}{4}\rho(Y;2|y_6|)(3)\\
&+ \frac{1}{2}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)(0)+\frac{1}{2}\lambda\left(Y;|y_3|+|y_6|,|\phi_3-\phi_6|\right)(0)\\
&+ \frac{1}{2}\lambda\left(Y;|y_5|+|y_6|,|\phi_5-\phi_6|\right)(-3)\\
&= \frac{2}{3}Y+\frac{1}{3}\rho(Y;2|y_3|)+\frac{1}{3}\rho(Y;2|y_4|)+\frac{3}{4}\rho(Y;2|y_5|)+\frac{3}{4}\rho(Y;2|y_6|)\\
&- \frac{3}{2}\lambda\left(Y;|y_5|+|y_6|,|\phi_5-\phi_6|\right)
\end{aligned}
$$

where the i and j indices on $\boldsymbol{\Gamma}_{2ij}$ specify the base tensor combinations.

For $\langle \boldsymbol{c}_1 \mid \boldsymbol{\Gamma}_{212} \mid \boldsymbol{c}_2\rangle$ the following inner products are needed:
$\left\langle \boldsymbol{c}_1 \mid \boldsymbol{T}_3^2,\boldsymbol{T}_3.\boldsymbol{T}_5,\boldsymbol{T}_3.\boldsymbol{T}_6,\boldsymbol{T}_5^2,\boldsymbol{T}_5.\boldsymbol{T}_6 \mid \boldsymbol{c}_2\right\rangle$

The $\boldsymbol{T}_3^2$ element is:

$$
\frac{1}{\sqrt{24}}\left(\delta_{ab}\delta_{kn}\right)^*\left(-t_{hlm}\right)\left(-t_{hmn}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) \quad = \quad 0
$$

as $\delta_{ba}d_{cab}=d_{caa}=0$

The $\boldsymbol{T}_5^2$ element is:

$$\frac{1}{\sqrt{24}}\left(\delta_{eb}\delta_{kl}\right)^{*}\left(-if_{hga}\right)\left(-if_{heg}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) = 0$$

as $\delta_{lk}t_{ckl} = t_{ckk} = 0$

The $\boldsymbol{T}_3.\boldsymbol{T}_5$ element is:

$$\begin{aligned}\frac{1}{\sqrt{24}}\left(\delta_{gb}\delta_{km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{hga}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) &= \frac{1}{\sqrt{24}}.\sqrt{\frac{3}{20}}if_{hga}d_{cga}t_{hlm}t_{cml} \\ &= 0\end{aligned}$$

as $if_{hga}d_{cga} = 0$

The $\boldsymbol{T}_3.\boldsymbol{T}_6$ element is:

$$\begin{aligned}\frac{1}{\sqrt{24}}\left(\delta_{ag}\delta_{km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{hgb}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) &= \frac{1}{\sqrt{24}}.\sqrt{\frac{3}{20}}if_{hab}d_{cab}t_{hlk}t_{ckl} \\ &= 0\end{aligned}$$

as $if_{hab}d_{cab} = 0$

The $\boldsymbol{T}_5.\boldsymbol{T}_6$ element is:

$$\frac{1}{\sqrt{24}}\left(\delta_{eg}\delta_{kl}\right)^{*}\left(-if_{hea}\right)\left(-if_{hgb}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) = 0$$

as $\delta_{lk} t_{ckl} = t_{ckk} = 0$

Thus:

$$\mathbf{\Gamma}_{212} \quad = \quad 0$$

For $\langle \boldsymbol{c}_1 \mid \mathbf{\Gamma}_{213} \mid \boldsymbol{c}_3 \rangle$ the following inner products are needed:
$\left\langle \boldsymbol{c}_1 \mid \boldsymbol{T}_3^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_3.\boldsymbol{T}_6, \boldsymbol{T}_5^2, \boldsymbol{T}_5.\boldsymbol{T}_6 \mid \boldsymbol{c}_3 \right\rangle$

The $\boldsymbol{T}_3^2$ element is:

$$\frac{1}{\sqrt{24}} (\delta_{ab}\delta_{kn})^* (-t_{hlm}) (-t_{hmn}) \frac{1}{\sqrt{12}} (i f_{cab} t_{ckl}) \quad = \quad 0$$

as $\delta_{ba} f_{cab} = f_{caa} = 0$

The $\boldsymbol{T}_5^2$ element is:

$$\frac{1}{\sqrt{24}} (\delta_{eb}\delta_{kl})^* \left(-i f_{hga}\right) \left(-i f_{heg}\right) \frac{1}{\sqrt{12}} (i f_{cab} t_{ckl}) \quad = \quad 0$$

as $\delta_{ba} f_{cab} = f_{caa} = 0$

The $\boldsymbol{T}_3.\boldsymbol{T}_5$ element is:

$$\begin{aligned}
\frac{1}{\sqrt{24}}\left(\delta_{eb}\delta_{km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{hea}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{1}{12\sqrt{2}}t_{hlm}f_{hea}f_{cae}t_{cml} \\
&= -\frac{1}{12\sqrt{2}}\left(\frac{\delta_{hc}}{2}\right)f_{hea}f_{cae} \\
&= -\frac{1}{24\sqrt{2}}f_{cea}f_{cae} \\
&= -\frac{1}{24\sqrt{2}}\left(-3\delta_{cc}\right) \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

The $\boldsymbol{T}_3.\boldsymbol{T}_6$ element is:

$$\begin{aligned}
\frac{1}{\sqrt{24}}\left(\delta_{ae}\delta_{km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{heb}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{1}{12\sqrt{2}}t_{hlm}f_{hab}f_{cab}t_{cml} \\
&= -\frac{1}{12\sqrt{2}}\left(\frac{\delta_{hc}}{2}\right)f_{hab}f_{cab} \\
&= -\frac{1}{24\sqrt{2}}f_{cab}f_{cab} \\
&= -\frac{1}{24\sqrt{2}}\left(3\delta_{cc}\right) \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}$$

The $\boldsymbol{T}_5.\boldsymbol{T}_6$ element is:

$$\frac{1}{\sqrt{24}}\left(\delta_{eg}\delta_{kl}\right)^{*}\left(-if_{hgb}\right)\left(-if_{hea}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) = 0$$

as $\delta_{kl}t_{ckl} = t_{ckk} = 0$.

Therefore

$$\begin{aligned}
\boldsymbol{\Gamma}_{213} &= \frac{1}{2}Y\left(2.\frac{1}{\sqrt{2}} - 2.\frac{1}{\sqrt{2}}\right) \\
&+ \frac{1}{2}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)\frac{1}{\sqrt{2}} + \frac{1}{2}\lambda\left(Y;|y_3|+|y_6|,|\phi_3-\phi_6|\right)\left(-\frac{1}{\sqrt{2}}\right) \\
&= \frac{1}{2\sqrt{2}}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right) - \frac{1}{2\sqrt{2}}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_6|\right)
\end{aligned}$$

For $\langle \boldsymbol{c}_2 \mid \boldsymbol{\Gamma}_{222} \mid \boldsymbol{c}_2 \rangle$ the following inner products are needed:
$\left\langle \boldsymbol{c}_2 \mid \boldsymbol{T}_3^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_3.\boldsymbol{T}_6, \boldsymbol{T}_5^2, \boldsymbol{T}_5.\boldsymbol{T}_6 \mid \boldsymbol{c}_2 \right\rangle$:

The $\boldsymbol{T}_3^2$ element is:

$$\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'ab}t_{c'kn}\right)^*\left(-t_{hlm}\right)\left(-t_{hmn}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) &= \frac{3}{20}d_{c'ba}t_{c'nk}t_{hlm}t_{hmn}d_{cab}t_{ckl} \\
&= \frac{3}{20}d_{c'ba}t_{c'nk}\left(\frac{4}{3}\delta_{ln}\right)d_{cab}t_{ckl} \\
&= \frac{1}{5}t_{c'lk}t_{ckl}d_{c'ba}d_{cab} \\
&= \frac{1}{5}\left(\frac{\delta_{cc'}}{2}\right)d_{c'ba}d_{cab} \\
&= \frac{1}{10}d_{cba}d_{cab} \\
&= \frac{1}{10}.\frac{5}{3}\delta_{cc} \\
&= \frac{4}{3}
\end{aligned}$$

The $\boldsymbol{T}_5^2$ element is:

$$
\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'eb}t_{c'kl}\right)^{*}\left(-if_{hga}\right)\left(-if_{heg}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) &= -\frac{3}{20}d_{c'be}t_{c'lk}f_{hga}f_{heg}d_{cab}t_{ckl} \\
&= -\frac{3}{20}d_{c'be}\left(\frac{\delta_{c'c}}{2}\right)f_{hga}f_{heg}d_{cab} \\
&= -\frac{3}{40}d_{cbe}f_{hga}f_{heg}d_{cab} \\
&= -\frac{3}{40}d_{cbe}\left(-f_{ahg}f_{ehg}\right)d_{cab} \\
&= -\frac{3}{40}d_{cbe}\left(-3\delta_{ae}\right)d_{cab} \\
&= \frac{9}{40}d_{cba}d_{cab} \\
&= \frac{9}{40}d_{cab}d_{cab} \\
&= \frac{9}{40}.\frac{5}{3}\delta_{cc} \\
&= 3
\end{aligned}
$$

The $\boldsymbol{T}_3.\boldsymbol{T}_5$ element is:

$$
\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'eb}t_{c'km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{hea}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) &= \frac{3i}{20}d_{c'be}t_{c'mk}t_{hlm}f_{hea}d_{cab}t_{ckl} \\
&= \frac{3i}{20}t_{c'mk}t_{hlm}t_{ckl}\left(\frac{5}{6}f_{hc'c}\right)
\end{aligned}
$$

as $f_{hea}d_{c'be}d_{cab} = \frac{5}{6}f_{hc'c}$

then

$$\frac{3i}{20}.\frac{5}{6}t_{c'mk}t_{hlm}t_{ckl}f_{hc'c} \quad = \quad \frac{1}{8}t_{ckl}\left(-\frac{3}{2}t_{clk}\right)$$

as $t_{hlm}t_{c'mk}\left(if_{hc'c}\right) = -\frac{3}{2}t_{clk}$

so

$$\begin{aligned} -\frac{3}{16}t_{ckl}t_{clk} &= -\frac{3}{16}.\frac{\delta_{cc}}{2} \\ &= -\frac{3}{4} \end{aligned}$$

The $\boldsymbol{T}_3.\boldsymbol{T}_6$ element is:

$$\begin{aligned} \sqrt{\frac{3}{20}}\left(d_{c'ae}t_{c'km}\right)^*\left(-t_{hlm}\right)\left(-if_{heb}\right)\sqrt{\frac{3}{20}}\left(d_{cab}t_{ckl}\right) &= \frac{3i}{20}d_{c'ea}t_{c'mk}t_{hlm}f_{heb}d_{cab}t_{ckl} \\ &= \frac{3i}{20}t_{c'mk}t_{hlm}t_{ckl}\left(-\frac{5}{6}f_{hcc'}\right) \end{aligned}$$

as $f_{heb}d_{cab}d_{c'ea} = -\frac{5}{6}f_{hcc'}$

then

$$-\frac{3i}{20}.\frac{5}{6}t_{c'mk}t_{hlm}t_{ckl}f_{hcc'} \quad = \quad -\frac{1}{8}t_{ckl}\left(\frac{3}{2}t_{clk}\right)$$

as $t_{hlm}t_{c'mk}if_{hcc'} = \frac{3}{2}t_{clk}$

so

$$
\begin{aligned}
-\frac{3}{16} t_{ckl} t_{clk} &= -\frac{3}{16} \cdot \frac{\delta_{cc}}{2} \\
&= -\frac{3}{4}
\end{aligned}
$$

The $\boldsymbol{T}_5.\boldsymbol{T}_6$ element is:

$$
\begin{aligned}
\sqrt{\frac{3}{20}} \left(d_{c'eg} t_{c'kl}\right)^* \left(-i f_{hea}\right) \left(-i f_{hgb}\right) \sqrt{\frac{3}{20}} \left(d_{cab} t_{ckl}\right) &= -\frac{3}{20} d_{c'ge} t_{c'lk} f_{hea} f_{hgb} d_{cab} t_{ckl} \\
&= -\frac{3}{20} d_{c'ge} \left(\frac{\delta_{cc'}}{2}\right) f_{hea} f_{hgb} d_{cab} \\
&= -\frac{3}{40} d_{cge} f_{hea} f_{hgb} d_{cab} \\
&= -\frac{3}{40} f_{eha} f_{hbg} d_{gce} d_{cab} \\
&= \frac{3}{40} f_{eah} f_{hbg} d_{gce} d_{cab} \\
&= -\frac{3}{40} \cdot \frac{3}{2} d_{abc} d_{cab}
\end{aligned}
$$

as $f_{eah} f_{hbg} d_{gce} = -\frac{3}{2} d_{abc}$

so

$$
\begin{aligned}
-\frac{9}{80} d_{abc} d_{cab} &= -\frac{9}{80} d_{abc} d_{abc} \\
&= -\frac{9}{80} \cdot \frac{5}{3} \delta_{aa} \\
&= -\frac{3}{2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\boldsymbol{\Gamma}_{222} &= \frac{1}{2}Y\left(\frac{4}{3}+2.3-2.\frac{3}{4}-2.\frac{3}{4}-2.\frac{3}{2}\right)+\frac{1}{4}\rho(Y;2|y_3|)\left(\frac{4}{3}\right)+\frac{1}{4}\rho(Y;2|y_4|)\left(\frac{4}{3}\right)\\
&+ \frac{1}{4}\rho(Y;2|y_5|).3+\frac{1}{4}\rho(Y;2|y_6|).3\\
&+ \frac{1}{2}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)\left(-\frac{3}{4}\right)+\frac{1}{2}\lambda\left(Y;|y_3|+|y_6|,|\phi_3-\phi_6|\right)\left(-\frac{3}{4}\right)\\
&+ \frac{1}{2}\lambda\left(Y;|y_5|+|y_6|,|\phi_5-\phi_6|\right)\left(-\frac{3}{2}\right)\\
&= \frac{2}{3}Y+\frac{1}{3}\rho(Y;2|y_3|)+\frac{1}{3}\rho(Y;2|y_4|)+\frac{3}{4}\rho(Y;2|y_5|)+\frac{3}{4}\rho(Y;2|y_6|)\\
&- \frac{3}{8}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)-\frac{3}{8}\lambda\left(Y;|y_3|+|y_6|,|\phi_3-\phi_6|\right)\\
&- \frac{3}{4}\lambda\left(Y;|y_5|+|y_6|,|\phi_5-\phi_6|\right)
\end{aligned}
$$

For $\langle \boldsymbol{c}_2 \mid \boldsymbol{\Gamma}_{223} \mid \boldsymbol{c}_3\rangle$ the following inner products are needed:
$\left\langle \boldsymbol{c}_2 \mid \boldsymbol{T}_3^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_3.\boldsymbol{T}_6, \boldsymbol{T}_5^2, \boldsymbol{T}_5.\boldsymbol{T}_6 \mid \boldsymbol{c}_3\right\rangle$

The $\boldsymbol{T}_3^2$ element is:

$$
\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'ab}t_{c'kn}\right)^*\left(-t_{hlm}\right)\left(-t_{hmn}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= \frac{i}{4\sqrt{5}}d_{c'ba}t_{c'nk}t_{hlm}t_{hmn}f_{cab}t_{ckl}\\
&= 0
\end{aligned}
$$

as $d_{c'ba}f_{cab} = d_{c'ab}f_{cab} = 0$

The $\boldsymbol{T}_5^2$ element is:

$$
\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'eb}t_{c'kl}\right)^{*}\left(-if_{hga}\right)\left(-if_{heg}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{i}{4\sqrt{5}}d_{c'be}t_{c'lk}f_{hga}f_{heg}f_{cab}t_{ckl} \\
&= -\frac{i}{4\sqrt{5}}d_{c'be}\left(\frac{\delta_{c'c}}{2}\right)f_{hga}f_{heg}f_{cab} \\
&= -\frac{i}{8\sqrt{5}}d_{cbe}f_{hga}f_{heg}f_{cab} \\
&= \frac{i}{8\sqrt{5}}d_{ecb}f_{acb}f_{hga}f_{heg} \\
&= 0
\end{aligned}
$$

as $d_{ecb}f_{acb}=0$

The $\boldsymbol{T}_3.\boldsymbol{T}_5$ element is:

$$
\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'eb}t_{c'km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{hea}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{1}{4\sqrt{5}}d_{c'be}t_{c'mk}t_{hlm}f_{hea}f_{cab}t_{ckl} \\
&= -\frac{1}{4\sqrt{5}}d_{c'be}f_{hea}f_{cab}\cdot\frac{1}{4}\left(if_{chc'}+d_{chc'}\right)
\end{aligned}
$$

as $t_{ckl}t_{hlm}t_{c'mk}=\frac{1}{4}\left(if_{chc'}+d_{chc'}\right)$

so

$$
-\frac{1}{16\sqrt{5}}d_{c'be}f_{hea}f_{cab}\left(if_{chc'}+d_{chc'}\right) = -\frac{1}{16\sqrt{5}}\left(id_{c'be}f_{hea}f_{cab}f_{chc'}+d_{c'be}f_{hea}f_{cab}d_{chc'}\right)
$$

dealing with the first part of the expression initially:

$$
\begin{aligned}
id_{c'be}f_{hea}f_{cab}f_{chc'} &= -id_{c'be}f_{hea}f_{abc}f_{cc'h} \\
&= -id_{c'be}\left(-\frac{3}{2}f_{ebc'}\right)
\end{aligned}
$$

as $f_{hea}f_{abc}f_{cc'h} = -\frac{3}{2}f_{ebc'}$

then

$$
\begin{aligned}
\frac{3i}{2}d_{c'be}f_{ebc'} &= \frac{3i}{2}d_{ebc'}f_{ebc'} \\
&= 0
\end{aligned}
$$

as $d_{ebc'}f_{ebc'} = 0$

dealing now with the second part of the expression

$$
\begin{aligned}
d_{c'be}f_{hea}f_{cab}d_{chc'} &= d_{c'be}f_{hea}f_{abc}d_{cc'h} \\
&= d_{c'be}\left(-\frac{3}{2}d_{ebc'}\right) \\
&= -\frac{3}{2}d_{c'be}d_{c'be} \\
&= -\frac{3}{2}.\frac{5}{3}\delta_{c'c'} \\
&= -20
\end{aligned}
$$

therefore

$$
\begin{aligned}
-\frac{1}{16\sqrt{5}}d_{c'be}f_{hea}f_{cab}\left(if_{chc'}+d_{chc'}\right) &= -\frac{1}{16\sqrt{5}}\left(-20\right)\\
&= \frac{5}{4\sqrt{5}}\\
&= \frac{\sqrt{5}}{4}
\end{aligned}
$$

The $\boldsymbol{T}_3.\boldsymbol{T}_6$ element is:

$$
\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'ae}t_{c'km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{heb}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{1}{4\sqrt{5}}d_{c'ea}t_{c'mk}t_{hlm}f_{heb}f_{cab}t_{ckl}\\
&= -\frac{1}{4\sqrt{5}}d_{c'ea}f_{heb}f_{cab}\cdot\frac{1}{4}\left(if_{c'ch}+d_{c'ch}\right)
\end{aligned}
$$

as $t_{ckl}t_{hlm}t_{c'mk}=\frac{1}{4}\left(if_{chc'}+d_{chc'}\right)$

so

$$
-\frac{1}{16\sqrt{5}}d_{c'ea}f_{heb}f_{cab}\left(if_{c'ch}+d_{c'ch}\right) = -\frac{1}{16\sqrt{5}}\left(id_{c'ea}f_{heb}f_{cab}f_{c'ch}+d_{c'ea}f_{heb}f_{cab}d_{c'ch}\right)
$$

dealing with the first part of the expression initially

$$
\begin{aligned}
id_{c'ea}f_{heb}f_{cab}f_{c'ch} &= -id_{c'ea}f_{heb}f_{bac}f_{cc'h}\\
&= -id_{c'ea}\left(-\frac{3}{2}f_{eac'}\right)
\end{aligned}
$$

as $f_{heb}f_{bac}f_{cc'h} = -\frac{3}{2}f_{eac'}$

then

$$\begin{aligned} \frac{3i}{2}d_{c'ea}f_{eac'} &= \frac{3i}{2}d_{c'ea}f_{c'ea} \\ &= 0 \end{aligned}$$

as $d_{c'ea}f_{c'ea} = 0$

dealing now with the second part of the expression

$$\begin{aligned} d_{c'ea}f_{heb}f_{cab}d_{c'ch} &= d_{c'ch}\left(-f_{ehb}f_{bca}d_{ac'e}\right) \\ &= d_{c'ch}\left(\frac{3}{2}d_{hcc'}\right) \\ &= \frac{3}{2}d_{hcc'}d_{hcc'} \\ &= \frac{3}{2}\left(\frac{5}{3}\delta_{hh}\right) \\ &= 20 \end{aligned}$$

therefore

$$\begin{aligned} -\frac{1}{16\sqrt{5}}d_{c'be}f_{hea}f_{cab}\left(if_{chc'} + d_{chc'}\right) &= -\frac{1}{16\sqrt{5}}(20) \\ &= -\frac{5}{4\sqrt{5}} \\ &= -\frac{\sqrt{5}}{4} \end{aligned}$$

The $\boldsymbol{T}_5.\boldsymbol{T}_6$ element is:

$$\begin{aligned}
\sqrt{\frac{3}{20}}\left(d_{c'eg}t_{c'kl}\right)^*\left(-if_{hea}\right)\left(-if_{hgb}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{i}{4\sqrt{5}}d_{c'ge}t_{c'lk}f_{hea}f_{hgb}f_{cab}t_{ckl} \\
&= -\frac{i}{4\sqrt{5}}d_{c'ge}\left(\frac{\delta_{c'c}}{2}\right)f_{hea}f_{hgb}f_{cab} \\
&= -\frac{i}{8\sqrt{5}}d_{cge}f_{hea}f_{hgb}f_{cab} \\
&= -\frac{i}{8\sqrt{5}}\left(\frac{3}{2}d_{cab}\right)f_{cab} \\
&= 0
\end{aligned}$$

as $d_{cge}f_{hea}f_{hgb} = \frac{3}{2}d_{cab}$ and $d_{cab}f_{cab} = 0$

Thus

$$\begin{aligned}
\boldsymbol{\Gamma}_{223} &= \frac{1}{2}Y(0+2.0+2.\frac{\sqrt{5}}{4}-2.\frac{\sqrt{5}}{4}+2.0) \\
&+ \frac{1}{4}\rho(Y;2\left|y_3\right|).0+\frac{1}{4}\rho(Y;2\left|y_4\right|).0+\frac{1}{4}\rho(Y;2\left|y_5\right|).0 \\
&+ \frac{1}{4}\rho(Y;2\left|y_6\right|).0+\frac{1}{2}\lambda\left(Y;\left|y_3\right|+\left|y_5\right|,\left|\phi_3-\phi_5\right|\right).\frac{\sqrt{5}}{4} \\
&+ \frac{1}{2}\lambda\left(Y;\left|y_3\right|+\left|y_6\right|,\left|\phi_3-\phi_6\right|\right)\left(-\frac{\sqrt{5}}{4}\right)+\frac{1}{2}\lambda\left(Y;\left|y_5\right|+\left|y_6\right|,\left|\phi_5-\phi_6\right|\right).0 \\
&= \frac{\sqrt{5}}{8}\lambda\left(Y;\left|y_3\right|+\left|y_5\right|,\left|\phi_3-\phi_5\right|\right)-\frac{\sqrt{5}}{8}\lambda\left(Y;\left|y_3\right|+\left|y_6\right|,\left|\phi_3-\phi_6\right|\right)
\end{aligned}$$

For $\langle \boldsymbol{c}_3 \mid \boldsymbol{\Gamma}_{233} \mid \boldsymbol{c}_3\rangle$ the following inner products are needed:
$\left\langle \boldsymbol{c}_3 \mid \boldsymbol{T}_3^2, \boldsymbol{T}_3.\boldsymbol{T}_5, \boldsymbol{T}_3.\boldsymbol{T}_6, \boldsymbol{T}_5^2, \boldsymbol{T}_5.\boldsymbol{T}_6 \mid \boldsymbol{c}_3\right\rangle$

The $\boldsymbol{T}_3^2$ element is:

$$\frac{1}{\sqrt{12}}\left(if_{c'ab}t_{c'kn}\right)^{*}\left(-t_{hlm}\right)\left(-t_{hmn}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right)$$

where $(if_{abc})^{*} = (if_{acb})$

therefore

$$\begin{aligned}
\frac{1}{\sqrt{12}}\left(if_{c'ab}t_{c'kn}\right)^{*}\left(-t_{hlm}\right)\left(-t_{hmn}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{1}{12}f_{c'ba}t_{c'nk}t_{hlm}t_{hmn}f_{cab}t_{ckl} \\
&= -\frac{1}{12}f_{c'ba}t_{c'nk}\left(\frac{4}{3}\delta_{ln}\right)f_{cab}t_{ckl} \\
&= -\frac{1}{9}f_{c'ba}t_{c'nk}f_{cab}t_{ckn} \\
&= -\frac{1}{9}f_{c'ba}\left(\frac{\delta_{c'c}}{2}\right)f_{cab} \\
&= -\frac{1}{18}f_{cba}f_{cab} \\
&= -\frac{1}{18}\left(-3\delta_{cc}\right) \\
&= \frac{4}{3}
\end{aligned}$$

The $\boldsymbol{T}_5^2$ element is:

$$
\begin{aligned}
\frac{1}{\sqrt{12}}\left(if_{c'eb}t_{c'kl}\right)^*\left(-if_{hga}\right)\left(-if_{heg}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= \frac{1}{12}f_{c'be}t_{c'lk}f_{hga}f_{heg}f_{cab}t_{ckl} \\
&= \frac{1}{12}f_{c'be}\left(\frac{\delta_{c'c}}{2}\right)f_{hga}f_{heg}f_{cab} \\
&= \frac{1}{24}f_{cbe}f_{hga}f_{heg}f_{cab} \\
&= -\frac{1}{24}f_{cbe}f_{ahg}f_{ehg}f_{cab} \\
&= -\frac{1}{24}f_{cbe}\left(3\delta_{ae}\right)f_{cab} \\
&= -\frac{1}{8}f_{cba}f_{cab} \\
&= -\frac{1}{8}\left(-3\delta_{cc}\right) \\
&= 3
\end{aligned}
$$

The $\boldsymbol{T}_3.\boldsymbol{T}_5$ element is:

$$
\begin{aligned}
\frac{1}{\sqrt{12}}\left(if_{c'eb}t_{c'km}\right)^*\left(-t_{hlm}\right)\left(-if_{hea}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{i}{12}f_{c'be}t_{c'mk}t_{hlm}f_{hea}f_{cab}t_{ckl} \\
&= -\frac{i}{12}\left(\frac{3}{2}f_{c'hc}\right)t_{c'mk}t_{hlm}t_{ckl}
\end{aligned}
$$

as $f_{c'be}f_{hea}f_{cab} = \frac{3}{2}f_{c'hc}$

then

$$
-\frac{i}{8}f_{c'hc}t_{c'mk}t_{hlm}t_{ckl} = -\frac{1}{8}\left(\frac{3}{2}t_{hml}\right)t_{hlm}
$$

as $t_{c'mk}t_{ckl}\left(if_{hc}\right)=\frac{3}{2}t_{hml}$

then

$$\begin{aligned}
-\frac{3}{16}t_{hml}t_{hlm} &= -\frac{3}{16}\left(\frac{\delta_{hh}}{2}\right) \\
&= -\frac{3}{4}
\end{aligned}$$

The $\boldsymbol{T}_3.\boldsymbol{T}_6$ element is:

$$\begin{aligned}
\frac{1}{\sqrt{12}}\left(if_{c'ae}t_{c'km}\right)^{*}\left(-t_{hlm}\right)\left(-if_{heb}\right)\frac{1}{\sqrt{12}}\left(if_{cab}t_{ckl}\right) &= -\frac{i}{12}f_{c'ea}t_{c'mk}t_{hlm}f_{heb}f_{cab}t_{ckl} \\
&= -\frac{i}{12}\left(-\frac{3}{2}f_{c'ch}\right)t_{c'mk}t_{hlm}t_{ckl}
\end{aligned}$$

as $f_{c'ea}f_{heb}f_{cab}=-\frac{3}{2}f_{c'ch}$

then

$$\frac{i}{8}f_{c'ch}t_{c'mk}t_{hlm}t_{ckl} = \frac{1}{8}\left(-\frac{3}{2}t_{hml}\right)t_{hlm}$$

as $t_{c'mk}t_{ckl}\left(if_{c'ch}\right)=-\frac{3}{2}t_{hml}$

then

$$\begin{aligned}
-\frac{3}{16}t_{hml}t_{hlm} &= -\frac{3}{16}\left(\frac{\delta_{hh}}{2}\right) \\
&= -\frac{3}{4}
\end{aligned}$$

The $\boldsymbol{T}_5.\boldsymbol{T}_6$ element is:

$$
\begin{aligned}
\frac{1}{\sqrt{12}}\left(if_{c'eg}t_{c'kl}\right)^*(-if_{hea})\left(-if_{hgb}\right)\frac{1}{\sqrt{12}}(if_{cab}t_{ckl}) &= \frac{1}{12}f_{c'ge}t_{c'lk}f_{hea}f_{hgb}f_{cab}t_{ckl} \\
&= \frac{1}{12}f_{c'ge}\left(\frac{\delta_{c'c}}{2}\right)f_{hea}f_{hgb}f_{cab} \\
&= \frac{1}{24}f_{cge}f_{hea}f_{hgb}f_{cab} \\
&= \frac{1}{24}f_{gce}f_{eah}f_{hbg}f_{cab} \\
&= \frac{1}{24}\left(-\frac{3}{2}f_{cab}\right)f_{cab} \\
&= -\frac{1}{16}(3\delta_{cc}) \\
&= -\frac{3}{2}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\mathbf{\Gamma}_{233} &= \frac{1}{2}Y(\frac{4}{3}+2.3+2\left(-\frac{3}{4}\right)+2\left(-\frac{3}{4}\right)+2\left(-\frac{3}{2}\right)) \\
&+ \frac{1}{4}\rho(Y;2\left|y_3\right|).\frac{4}{3}+\frac{1}{4}\rho(Y;2\left|y_4\right|).\frac{4}{3} \\
&+ \frac{1}{4}\rho(Y;2\left|y_5\right|).3+\frac{1}{4}\rho(Y;2\left|y_6\right|).3 \\
&+ \frac{1}{2}\lambda\left(Y;\left|y_3\right|+\left|y_5\right|,\left|\phi_3-\phi_5\right|\right)\left(-\frac{3}{4}\right) \\
&+ \frac{1}{2}\lambda\left(Y;\left|y_3\right|+\left|y_6\right|,\left|\phi_3-\phi_6\right|\right)\left(-\frac{3}{4}\right)+\frac{1}{2}\lambda\left(Y;\left|y_5\right|+\left|y_6\right|,\left|\phi_5-\phi_6\right|\right)\left(-\frac{3}{2}\right) \\
&= \frac{2}{3}Y+\frac{1}{3}\rho(Y;2\left|y_3\right|)+\frac{1}{3}\rho(Y;2\left|y_4\right|)+\frac{3}{4}\rho(Y;2\left|y_5\right|)+\frac{3}{4}\rho(Y;2\left|y_6\right|) \\
&- \frac{3}{8}\lambda\left(Y;\left|y_3\right|+\left|y_5\right|,\left|\phi_3-\phi_5\right|\right)-\frac{3}{8}\lambda\left(Y;\left|y_3\right|+\left|y_6\right|,\left|\phi_3-\phi_6\right|\right) \\
&- \frac{3}{4}\lambda\left(Y;\left|y_5\right|+\left|y_6\right|,\left|\phi_5-\phi_6\right|\right)
\end{aligned}
$$

The ADM is clearly symmetric and thus the full matrix for two gluons emitted on the left $\mathbf{\Gamma}_{2L}$ is:

$$
\begin{pmatrix}
\begin{array}{c}\frac{2}{3}Y+\frac{1}{3}\rho_3+\frac{1}{3}\rho_4\\ +\frac{3}{4}\rho_5+\frac{3}{4}\rho_6\\ -\frac{3}{2}\lambda_{56}\end{array} & 0 & \frac{1}{2\sqrt{2}}\lambda_{35}-\frac{1}{2\sqrt{2}}\lambda_{36} \\
0 & \begin{array}{c}\frac{2}{3}Y+\frac{1}{3}\rho_3+\frac{1}{3}\rho_4\\ +\frac{3}{4}\rho_5+\frac{3}{4}\rho_6-\frac{3}{8}\lambda_{35}\\ -\frac{3}{8}\lambda_{36}-\frac{3}{4}\lambda_{56}\end{array} & \frac{\sqrt{5}}{8}\lambda_{35}-\frac{\sqrt{5}}{8}\lambda_{36} \\
\frac{1}{2\sqrt{2}}\lambda_{35}-\frac{1}{2\sqrt{2}}\lambda_{36} & \frac{\sqrt{5}}{8}\lambda_{35}-\frac{\sqrt{5}}{8}\lambda_{36} & \begin{array}{c}\frac{2}{3}Y+\frac{1}{3}\rho_3+\frac{1}{3}\rho_4\\ +\frac{3}{4}\rho_5+\frac{3}{4}\rho_6\\ -\frac{3}{8}\lambda_{35}-\frac{3}{8}\lambda_{36}-\frac{3}{4}\lambda_{56}\end{array}
\end{pmatrix}
$$

where the arguments have been omitted for clarity.

3.2 Out Of Gap Gluon Emission Matrices

The $\mathbf{\Gamma}$ matrices calculated above add the virtual in- gap gluon dressing to the amplitude on which they act. For the zero gluons outside the gap case they are the only element needed. For the case of one and two gluons outside the gap it is necessary to calculate the out of gap gluon emission matrices. The out of gap gluon emissions matrices may either add a real gluon $\boldsymbol{D}_{na\mu}$(where n is the out of gap gluon, a is its colour index and μ is the Lorentz index) or an eikonal $\boldsymbol{\gamma}$ (real part of a virtual gluon - after integrating over rapidity and azimuth) gluon to a basis, the $\mathbf{\Gamma}$ matrices then dress the result. Using the appropriate sequence of $\boldsymbol{D}$, $\boldsymbol{\gamma}$ and $\mathbf{\Gamma}$ matrices allows the corrections from emission and subsequent in gap dressing of any number of out of gap real gluons.

3.2.1 The $\boldsymbol{D}$ Matrices

Each $\boldsymbol{D}_{na\mu}$ matrix increases the dimensionality of the colour space on which it acts by one [5]. The bra basis tensor must therefore have one more gluon index than the ket basis tensor. Two $\boldsymbol{D}_{na\mu}$ matrices need to be calculated. $\boldsymbol{D}_{0a\mu}$ adds a real gluon to $q\overline{q}$ whilst $\boldsymbol{D}_{1a\mu}$ adds a (second) real gluon to $q\overline{q}g$.

$\boldsymbol{D}_{0a\mu}$

The$\boldsymbol{D}_{0a\mu}$ matrix calculation is $\langle q\overline{q}g \mid \boldsymbol{D}_{0a\mu} \mid q\overline{q}\rangle$. The $q\overline{q}$ basis $\boldsymbol{c}$ is $\frac{1}{\sqrt{3}}\delta_{kl}$ (by colour conservation) whilst that of $q\overline{q}g, \boldsymbol{c}^{'}$ is $\frac{1}{2}t_{akl}$.

$\boldsymbol{D}_{0a\mu}$ is given by [5]:

$$\sum_i \boldsymbol{T}_{ia} h_{i\mu} \tag{3.2}$$

where:

$$h_{i\mu} = \frac{1}{2} k_T \frac{p_{i\mu}}{p_i . k} \tag{3.3}$$

and k_T is the transverse momentum of the emitted gluon, $p_{i\mu}$ is the 4 momentum of the i^{th} emitting parton and k is the 4 momentum of the emitted gluon.

Therefore identifying $\bar{q}$ and q as partons 3 and 4 respectively and the gluon as parton 5, means the colour calculations $\left\langle \boldsymbol{c}^{'} \mid -\boldsymbol{t}_a \mid \boldsymbol{c} \right\rangle$ and $\left\langle \boldsymbol{c}^{'} \mid \boldsymbol{t}_a \mid \boldsymbol{c} \right\rangle$ are needed.

For $\left\langle \boldsymbol{c}^{'} \mid -\boldsymbol{t}_a \mid \boldsymbol{c} \right\rangle$ the calculation is (see Figure 3.4):

$$\begin{aligned} \left(\frac{1}{2} t_{akm}\right)^* (-t_{alm}) \left(\frac{1}{\sqrt{3}} \delta_{kl}\right) &= -\frac{1}{2\sqrt{3}} t_{aml} t_{alm} \\ &= -\frac{1}{2\sqrt{3}} \left(\frac{\delta_{aa}}{2}\right) \\ &= -\frac{2}{\sqrt{3}} \end{aligned}$$

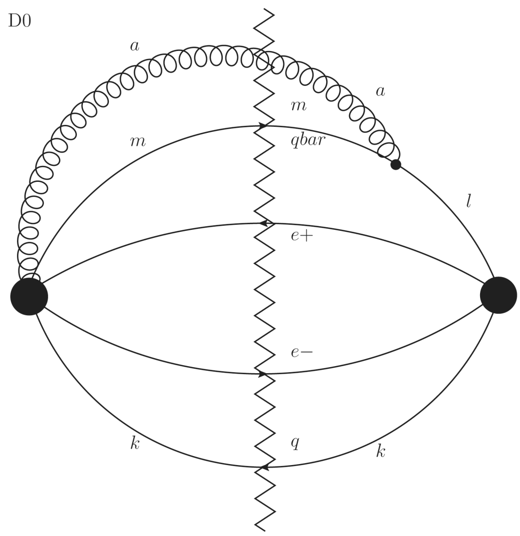

Figure 3.4: Primary Out Of Gap Real Emission

And $\left\langle \boldsymbol{c}' \mid \boldsymbol{t}_a \mid \boldsymbol{c} \right\rangle$

$$\begin{aligned}\left(\frac{1}{2}t_{aml}\right)^* t_{amk}\left(\frac{1}{\sqrt{3}}\delta_{kl}\right) &= \frac{1}{2\sqrt{3}}t_{akm}t_{amk} \\ &= \frac{1}{2\sqrt{3}}\left(\frac{\delta_{aa}}{2}\right) \\ &= \frac{2}{\sqrt{3}}\end{aligned}$$

Therefore

$$\boldsymbol{D}_{0a\mu} = \frac{2}{\sqrt{3}}\left(h_{4\mu} - h_{3\mu}\right) \tag{3.4}$$

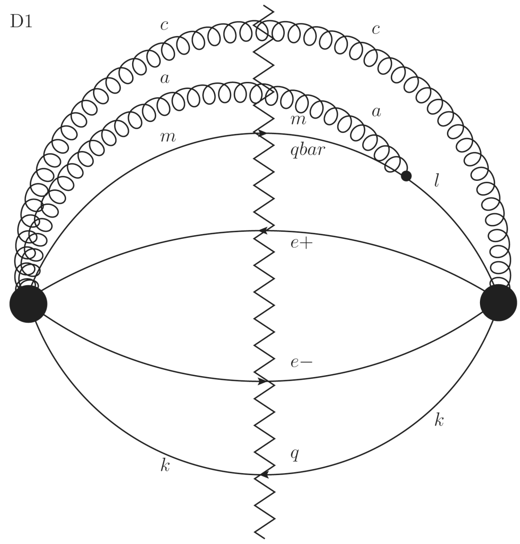

Figure 3.5: Secondary Out Of Gap Real Emission

$\boldsymbol{D}_{1a\mu}$

The $\boldsymbol{D}_{1a\mu}$ matrix calculation is $\left\langle q\overline{q}gg \mid \boldsymbol{D}_{1a\mu} \mid q\overline{q}g \right\rangle$. The $q\overline{q}g$ basis (now redesignated $\boldsymbol{c}$) is as above whilst there are now three basis vectors for $q\overline{q}gg$, (as used previously in the two gluon outside the gap calculations) $\boldsymbol{c}'_1 = \frac{1}{\sqrt{24}}\delta_{ab}\delta_{kl}$, $\boldsymbol{c}'_2 = \sqrt{\frac{3}{20}}d_{c'ab}t_{c'kl}$ and $\boldsymbol{c}'_3 = \frac{1}{\sqrt{12}}if_{c'ab}t_{c'kl}$. $\boldsymbol{T}_3$ and $\boldsymbol{T}_4$ are the antiquark and quark operators as above, whilst $\boldsymbol{T}_5 = -i\boldsymbol{f}_a$.

$\boldsymbol{D}_{1a\mu}$ is thus the 3.1 matrix $\begin{pmatrix} \langle \boldsymbol{c}'_1 \mid \boldsymbol{T}_3 + \boldsymbol{T}_4 + \boldsymbol{T}_5 \mid \boldsymbol{c} \rangle \\ \langle \boldsymbol{c}'_2 \mid \boldsymbol{T}_3 + \boldsymbol{T}_4 + \boldsymbol{T}_5 \mid \boldsymbol{c} \rangle \\ \langle \boldsymbol{c}'_3 \mid \boldsymbol{T}_3 + \boldsymbol{T}_4 + \boldsymbol{T}_5 \mid \boldsymbol{c} \rangle \end{pmatrix}$

The following results are therefore needed: $\langle \boldsymbol{c}'_1 \mid -\boldsymbol{t}_a \mid \boldsymbol{c} \rangle$ (see Figure 3.5)

$$\begin{aligned}
\frac{1}{\sqrt{24}}(\delta_{ac}\delta_{km})^*(-t_{alm})\frac{1}{2}(t_{ckl}) &= -\frac{1}{2\sqrt{24}}t_{alk}t_{akl} \\
&= -\frac{1}{2\sqrt{24}}\left(\frac{\delta_{aa}}{2}\right) \\
&= -\frac{1}{\sqrt{6}}
\end{aligned}$$

$\langle \boldsymbol{c}_1' \mid \boldsymbol{t}_a \mid \boldsymbol{c}\rangle$

$$\begin{aligned}
\frac{1}{\sqrt{24}}(\delta_{ac}\delta_{ml})^* t_{amk}\frac{1}{2}(t_{ckl}) &= \frac{1}{2\sqrt{24}}t_{alk}t_{akl} \\
&= \frac{1}{2\sqrt{24}}\left(\frac{\delta_{aa}}{2}\right) \\
&= \frac{1}{\sqrt{6}}
\end{aligned}$$

$\langle \boldsymbol{c}_1' \mid -i\boldsymbol{f}_a \mid \boldsymbol{c}\rangle$

$$\begin{aligned}
\frac{1}{\sqrt{24}}(\delta_{ab}\delta_{kl})^*(-if_{abc})\frac{1}{2}(t_{ckl}) &= -\frac{i}{2\sqrt{24}}f_{aac}t_{cll} \\
&= 0
\end{aligned}$$

as both f_{aac} and t_{cll} are zero.

$\langle \boldsymbol{c}_2' \mid -\boldsymbol{t}_a \mid \boldsymbol{c}\rangle$

$$\begin{aligned}\sqrt{\frac{3}{20}}\left(d_{c'ac}t_{c'km}\right)^{*}(-t_{alm})\frac{1}{2}(t_{ckl}) &= -\frac{1}{2}\sqrt{\frac{3}{20}}d_{c'ca}t_{c'mk}t_{alm}t_{ckl}\\ &= -\frac{5}{12}\sqrt{\frac{3}{20}}t_{alm}t_{aml}\end{aligned}$$

as $t_{c'mk}t_{ckl}d_{c'ca} = \frac{5}{6}t_{aml}$

$$\begin{aligned}-\frac{5}{12}\sqrt{\frac{3}{20}}t_{alm}t_{aml} &= -\frac{5}{12}\sqrt{\frac{3}{20}}\left(\frac{\delta_{aa}}{2}\right)\\ &= -\sqrt{\frac{5}{12}}\end{aligned}$$

$\langle \boldsymbol{c}_2' \mid \boldsymbol{t}_a \mid \boldsymbol{c}\rangle$

$$\sqrt{\frac{3}{20}}\left(d_{c'ac}t_{c'ml}\right)^{*}t_{amk}\frac{1}{2}(t_{ckl}) = \frac{1}{2}\sqrt{\frac{3}{20}}d_{c'ca}t_{c'lm}t_{amk}t_{ckl}$$

contracted as above, therefore

$$\begin{aligned}\frac{1}{2}\sqrt{\frac{3}{20}}d_{c'ca}t_{c'lm}t_{amk}t_{ckl} &= \frac{5}{12}\sqrt{\frac{3}{20}}t_{clk}t_{ckl}\\ &= \frac{5}{12}\sqrt{\frac{3}{20}}\left(\frac{\delta_{cc}}{2}\right)\\ &= \sqrt{\frac{5}{12}}\end{aligned}$$

$\langle \boldsymbol{c}_2' \mid -i\boldsymbol{f}_a \mid \boldsymbol{c}\rangle$

$$\begin{aligned}\sqrt{\frac{3}{20}}\left(d_{c'ab}t_{c'kl}\right)^{*}\left(-if_{abc}\right)\frac{1}{2}\left(t_{ckl}\right) &= -\frac{i}{2}\sqrt{\frac{3}{20}}d_{c'ba}f_{abc}t_{c'lk}t_{ckl} \\ &= 0\end{aligned}$$

as $d_{c'ba}f_{abc} = d_{c'ab}f_{cab} = 0$.

$\left\langle \boldsymbol{c}'_3 \mid -\boldsymbol{t}_a \mid \boldsymbol{c} \right\rangle$

$$\begin{aligned}\frac{1}{\sqrt{12}}\left(if_{c'ac}t_{c'km}\right)^{*}\left(-t_{alm}\right)\frac{1}{2}\left(t_{ckl}\right) &= -\frac{i}{2\sqrt{12}}f_{c'ca}t_{c'mk}t_{ckl}t_{alm} \\ &= \frac{1}{2\sqrt{12}}.\frac{3}{2}t_{aml}t_{alm}\end{aligned}$$

as $-if_{c'ca}t_{c'mk}t_{ckl} = \frac{3}{2}t_{aml}$

$$\begin{aligned}\frac{3}{4\sqrt{12}}t_{aml}t_{alm} &= \frac{3}{4\sqrt{12}}\left(\frac{\delta_{aa}}{2}\right) \\ &= \frac{3}{\sqrt{12}}\end{aligned}$$

$\left\langle \boldsymbol{c}'_3 \mid \boldsymbol{t}_a \mid \boldsymbol{c} \right\rangle$

$$\frac{1}{\sqrt{12}}\left(if_{c'ac}t_{c'ml}\right)^{*}t_{amk}\frac{1}{2}\left(t_{ckl}\right) = \frac{i}{2\sqrt{12}}f_{c'ca}t_{c'lm}t_{amk}t_{ckl}$$

contracted as above, therefore

$$\begin{aligned}
\frac{i}{2\sqrt{12}} f_{c'ca} t_{c'lm} t_{amk} t_{ckl} &= \frac{3}{4\sqrt{12}} t_{amk} t_{akm} \\
&= \frac{3}{4\sqrt{12}} \left(\frac{\delta_{aa}}{2}\right) \\
&= \frac{3}{\sqrt{12}}
\end{aligned}$$

$\left\langle \boldsymbol{c}_3' \mid -i\boldsymbol{f}_a \mid \boldsymbol{c} \right\rangle$

$$\begin{aligned}
\frac{1}{\sqrt{12}} \left(if_{c'ab} t_{c'kl}\right)^* \left(-if_{abc}\right) \frac{1}{2} \left(t_{ckl}\right) &= \frac{1}{2\sqrt{12}} f_{c'ba} t_{c'lk} f_{abc} t_{ckl} \\
&= -\frac{1}{2\sqrt{12}} \left(3\delta_{c'c}\right) t_{c'lk} t_{ckl}
\end{aligned}$$

as $f_{c'ba} f_{abc} = -f_{c'ab} f_{cab} = 3\delta_{c'c}$, therefore

$$\begin{aligned}
-\frac{1}{2\sqrt{12}} .3\delta_{c'c} t_{c'lk} t_{ckl} &= -\frac{3}{2\sqrt{12}} t_{clk} t_{ckl} \\
&= -\frac{3}{2\sqrt{12}} \left(\frac{\delta_{cc}}{2}\right) \\
&= -\frac{\sqrt{12}}{2}
\end{aligned}$$

Therefore

$$\boldsymbol{D}_{1a\mu} = \begin{pmatrix} \frac{1}{\sqrt{6}}h_{4\mu} - \frac{1}{\sqrt{6}}h_{3\mu} \\ \sqrt{\frac{5}{12}}h_{4\mu} - \sqrt{\frac{5}{12}}h_{3\mu} \\ \frac{3}{\sqrt{12}}h_{4\mu} + \frac{3}{\sqrt{12}}h_{3\mu} - \frac{\sqrt{12}}{2}h_{5\mu} \end{pmatrix} \tag{3.5}$$

3.2.2 The $\boldsymbol{\gamma}$ Matrices

Two $\boldsymbol{\gamma}$ matrices need to be calculated.$\boldsymbol{\gamma}_0$ adds a virtual (eikonal) out of gap gluon to the $q\bar{q}$ final state whilst $\boldsymbol{\gamma}_1$ adds a virtual (eikonal) out of gap gluon to $q\bar{q}g$ [5].

$$\boldsymbol{\gamma} \quad = \quad -\frac{1}{2}\sum_{i<j}\boldsymbol{T}_i.\boldsymbol{T}_j\omega_{ij}$$

where:

$$\omega_{ij} = \tfrac{1}{2}h_{i\mu}.h_{j\mu} \quad = \frac{1}{2}k_T^2\frac{p_i.p_j}{(p_i.k)\left(p_j.k\right)}$$

For $\boldsymbol{\gamma}_0$ therefore, only the $\boldsymbol{T}_3.\boldsymbol{T}_4$ operator combination is needed. As the virtual gluons do not change the dimensionality of the colour space the basis tensor in the bra and ket is the same and so the colour element required is $\left\langle \boldsymbol{c} \mid -\frac{1}{2}\left(-\boldsymbol{t}_a.\boldsymbol{t}_a\right) \mid \boldsymbol{c}\right\rangle$ (see Figure 3.6) where $\boldsymbol{c} = \frac{1}{\sqrt{3}}\delta_{kl}$.

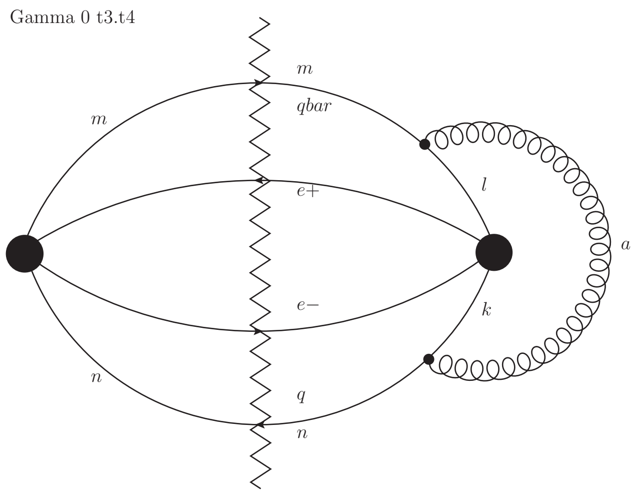

Figure 3.6: Calculating $\boldsymbol{\gamma}_0$

$$
\begin{aligned}
\left\langle \boldsymbol{c} \mid -\frac{1}{2}\left(-\boldsymbol{t}_a.\boldsymbol{t}_a\right) \mid \boldsymbol{c} \right\rangle &= \frac{1}{\sqrt{3}}\left(\delta_{nm}\right)^{*}\left(-t_{alm}\right) t_{ank}\frac{1}{\sqrt{3}}\left(\delta_{kl}\right) \\
&= -\frac{1}{3} t_{alm} t_{aml} \\
&= -\frac{1}{3}\left(\frac{\delta_{aa}}{2}\right) \\
&= -\frac{4}{3}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\boldsymbol{\gamma}_0 &= -\frac{1}{2}\left(-\frac{4}{3}\right)\omega_{34} \\
&= \frac{2}{3}\omega_{34}
\end{aligned}
$$

For the $\boldsymbol{\gamma}_1$ matrix the $\boldsymbol{T}_3.\boldsymbol{T}_4, \boldsymbol{T}_3.\boldsymbol{T}_5$ and $\boldsymbol{T}_4.\boldsymbol{T}_5$ colour operator combinations are needed. The colour element required is therefore
$\left\langle \boldsymbol{c} \mid -\frac{1}{2}\left(-\boldsymbol{t}_a.\boldsymbol{t}_a + (-\boldsymbol{t}_a)(-i\boldsymbol{f}_a) + \boldsymbol{t}_a(-i\boldsymbol{f}_a)\right) \mid \boldsymbol{c} \right\rangle$, where $\boldsymbol{c} = \frac{1}{2} t_{ckl}$. The $\boldsymbol{T}_3.\boldsymbol{T}_5$ combination has already been calculated for $\boldsymbol{\Gamma}_1$ one gluon outside the gap and is $-\frac{3}{2}$.

The $\boldsymbol{T}_4.\boldsymbol{T}_5$ calculation is:

$$\begin{aligned}
\langle \boldsymbol{c} \mid \boldsymbol{t}_a.(-i\boldsymbol{f}_a) \mid \boldsymbol{c} \rangle &= \frac{1}{2}(t_{eml})^* t_{hmk}(-if_{hec})\frac{1}{2}(t_{ckl}) \\
&= \frac{1}{4} t_{elm} t_{hmk} t_{ckl} i f_{hce} \\
&= \frac{1}{4} t_{elm}\left(-\frac{3}{2} t_{eml}\right) \\
&= -\frac{3}{8} t_{elm} t_{ekm} \\
&= -\frac{3}{8}\left(\frac{\delta_{ee}}{2}\right) \\
&= -\frac{3}{2}
\end{aligned}$$

The $\boldsymbol{T}_3.\boldsymbol{T}_4$ calculation is:

$$\begin{aligned}
\langle \boldsymbol{c} \mid -\boldsymbol{t}_a.\boldsymbol{t}_a \mid \boldsymbol{c} \rangle &= \left(\frac{1}{2} t_{cnm}\right)^* (-t_{alm}) t_{ank} \left(\frac{1}{2} t_{ckl}\right) \\
&= -\frac{1}{4} t_{cmn} t_{ckl} t_{alm} t_{ank} \\
&= -\frac{1}{4}\left(-\frac{1}{6} t_{aml}\right) t_{alm}
\end{aligned}$$

as $t_{cmn} t_{ank} t_{ckl} = -\frac{1}{6} t_{aml}$, therefore

$$
\begin{aligned}
\frac{1}{24} t_{aml} t_{alm} &= \frac{1}{24}\left(\frac{\delta_{aa}}{2}\right) \\
&= \frac{1}{6}
\end{aligned}
$$

therefore:

$$
\begin{aligned}
\boldsymbol{\gamma}_1 &= -\frac{1}{2}\left(-\frac{3}{2}\omega_{35} - \frac{3}{2}\omega_{45} + \frac{1}{6}\omega_{34}\right) \\
&= \frac{3}{4}\left(\omega_{35} + \omega_{45}\right) - \frac{1}{12}\omega_{34}
\end{aligned}
$$

3.3 Zero, One And Two Gluon Out Of The Gap Cross-Sections

Having calculated the matrices for adding real and virtual gluons outside the gap and the virtual in gap dressing of these expressions, it is now possible to calculate the cross-sections.

3.3.1 Zero Gluons Outside The Gap

The amplitude $\boldsymbol{M}$ for zero gluons outside the gap is [7]:

$$\begin{aligned}\mathbf{M} &= \exp\left(-\frac{2}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\alpha_s\mathbf{\Gamma}_0\right)\boldsymbol{M}_0 \\ &= \exp\left(-\frac{2}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\alpha_s\left(\frac{2}{3}Y+\frac{1}{3}\rho\left(Y,2\left|y_3\right|\right)+\frac{1}{3}\rho\left(Y,2\left|y_4\right|\right)\right)\right)\boldsymbol{M}_0\end{aligned}$$

where $\boldsymbol{M}_0$ is the Born amplitude. The limits of integration correspond to the in-gap region at transverse momenta above Q_0 and thus capture the virtual dressing.

As

$$2\left|y_3\right| = 2\left|y_4\right| = \Delta y$$

[7]

then

$$\mathbf{\Gamma}_0 = \left(\frac{2}{3}Y+\frac{2}{3}\rho\left(Y,\left|\Delta y\right|\right)\right)$$

and

$$-\frac{2}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\alpha_s\left(\frac{2}{3}Y+\frac{2}{3}\rho\left(Y,\left|\Delta y\right|\right)\right) = -\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\left(\frac{4}{3}Y+\frac{4}{3}\rho\left(Y,\left|\Delta y\right|\right)\right)$$

The cross-section σ_0=$\boldsymbol{M}^{\dagger}\boldsymbol{M}$. Expanding $\boldsymbol{M}$ as a perturbation series in powers of α_s, with $\mathbf{\Gamma}_0 = \left(\frac{4}{3}Y+\frac{4}{3}\rho\left(Y,\left|\Delta y\right|\right)\right)$, to $O\left(\alpha_s\right)^3$ gives:

$$\boldsymbol{M} = \left(1-\left(-\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\Gamma_0\right)+\frac{1}{2}\left(-\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\Gamma_0\right)^2-\frac{1}{6}\left(-\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\Gamma_0\right)^3\right)\boldsymbol{M}_0$$

if

$$a = -\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\Gamma_0$$

then

$$\boldsymbol{M} = \left(1+a+\frac{1}{2}a^2+\frac{1}{6}a^3\right)\boldsymbol{M}_0$$

so

$$\sigma_0 = \left(1+a+\frac{1}{2}a^2+\frac{1}{6}a^3\right)^\dagger\left(1+a+\frac{1}{2}a^2+\frac{1}{6}a^3\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0$$

as $\boldsymbol{\Gamma}_0^\dagger = \boldsymbol{\Gamma}_0$, then $a^\dagger = a$, so

$$\begin{aligned}\sigma_0 &= \left(1+a+\frac{1}{2}a^2+\frac{1}{6}a^3\right)^2\boldsymbol{M}_0^\dagger\boldsymbol{M}_0\\ &= \left(1+2a+2a^2+\frac{4}{3}a^3\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0\end{aligned}$$

to $O(\alpha_s)^3$.

3.3.2 One Gluon Outside The Gap

The cross-section for one gluon outside the gap is composed of two elements one real Ω_R and one virtual Ω_V.

$$\sigma_1 = -\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\int_{out}\frac{dyd\phi}{2\pi}\left(\Omega_R+\Omega_V\right)$$

The explicit integral is over the phase space for the out of gap gluon. Ω_R involves the global dressing of the quark-antiquark pair ($\mathbf{\Gamma}_0$) at transverse momenta from Q to k_T followed by the emission of a real out of gap gluon $\boldsymbol{D}_{0\mu}$ at transverse momentum k_T and then the subsequent non-global virtual dressing of these three partons by $\mathbf{\Gamma}_1$ from transverse momenta k_T to Q_0 (see the left hand frame of Figure 3.7 where the secondary virtual gluon is understood to dress the three real partons in all possible ways).

$$\begin{aligned}\Omega_R = \; & \exp\left(-\frac{2\alpha_s}{\pi}\int_{k_T}^{Q}\frac{dk'_T}{k'_T}\mathbf{\Gamma}_0^\dagger\right)\boldsymbol{D}_{0\mu}^\dagger\exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\mathbf{\Gamma}_1^\dagger\right)\\ & \exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\mathbf{\Gamma}_1\right)\boldsymbol{D}_{0\mu}\exp\left(-\frac{2\alpha_s}{\pi}\int_{k_T}^{Q}\frac{dk'_T}{k'_T}\mathbf{\Gamma}_0\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0\end{aligned}$$

with the k'_T integral being over the in-gap dressing.

Ω_V also involves the global dressing of the quark-antiquark pair ($\mathbf{\Gamma}_0$) at transverse momenta from Q to k_T followed by the emission of a virtual (eikonal) out of gap gluon $\boldsymbol{\gamma}_0$ at transverse momentum k_T and then the subsequent non-global

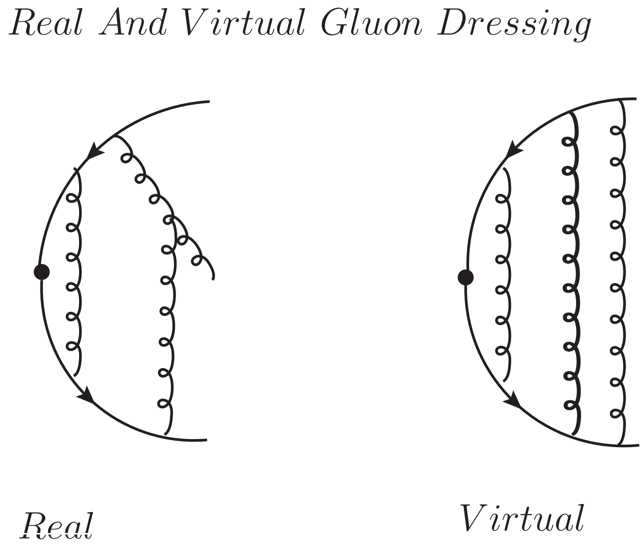

Figure 3.7: Dressing Of One Gluon Outside The Gap

virtual dressing of these three partons (though there are no virtual emissions from the eikonal gluon) by $\mathbf{\Gamma}_0$ from transverse momenta k_T to Q_0 (see the right hand frame of Figure 3.7 where the darker virtual gluon in the virtual frame represents the eikonal emission).

$$\Omega_V = \boldsymbol{M}_0^\dagger \boldsymbol{M}_0 \exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk'_T}{k'_T}\mathbf{\Gamma}_0^\dagger\right)\exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\mathbf{\Gamma}_0\right)\boldsymbol{\gamma}_0$$
$$\exp\left(-\frac{2\alpha_s}{\pi}\int_{k_T}^{Q}\frac{dk'_T}{k'_T}\mathbf{\Gamma}_0\right) + \text{complex conjugate (c.c.)}$$

The presence of a power of α_s for the out of gap gluon means that the in-gap dressings only need be expanded to $O(\alpha_s)^2$, to have an overall expression to $O(\alpha_s)^3$.

Rearranging Ω_R gives:

$$\boldsymbol{D}_{0\mu}^{\dagger}\boldsymbol{D}_{0\mu}\boldsymbol{M}_0^{\dagger}\boldsymbol{M}_0 \exp\left(-\tfrac{2\alpha_s}{\pi}\int_{k_T}^{Q}\tfrac{dk'_T}{k'_T}\boldsymbol{\Gamma}_0^{\dagger}\right)$$
$$\exp\left(-\tfrac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\tfrac{dk'_T}{k'_T}\boldsymbol{\Gamma}_1^{\dagger}\right)\exp\left(-\tfrac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\tfrac{dk'_T}{k'_T}\boldsymbol{\Gamma}_1\right)\exp\left(-\tfrac{2\alpha_s}{\pi}\int_{k_T}^{Q}\tfrac{dk'_T}{k'_T}\boldsymbol{\Gamma}_0\right)$$

now

$$\begin{aligned}
\boldsymbol{D}_{0\mu}^{\dagger}\boldsymbol{D}_{0\mu} &= \frac{2}{\sqrt{3}}\left(h_{4\mu}-h_{3\mu}\right)^{\dagger}\frac{2}{\sqrt{3}}\left(h_{4\mu}-h_{3\mu}\right) \\
&= \frac{4}{3}\left(h_{4\mu}^2 - h_{4\mu}h_{3\mu} - h_{3\mu}h_{4\mu} + h_{3\mu}^2\right) \\
&= -\frac{4}{3}\left(h_{4\mu}h_{3\mu} + h_{3\mu}h_{4\mu}\right) \\
&= -\frac{4}{3}\left(2h_{3\mu}h_{4\mu}\right) \\
&= -\frac{8}{3}h_{3\mu}h_{4\mu} \\
&= -\frac{4}{3}\omega_{34}
\end{aligned}$$

as $h_{3\mu}^2 = h_{4\mu}^2 = 0$ for massless particles and $\omega_{ij} \equiv 2h_i.h_j$.

Dealing with the exponents, noting that $\boldsymbol{\Gamma}_0^{\dagger} = \boldsymbol{\Gamma}_0$ and $\boldsymbol{\Gamma}_1^{\dagger} = \boldsymbol{\Gamma}_1$ and performing the k'_T integral, leads to:

$$\exp\left(-\frac{2\alpha_s}{\pi}\ln\frac{Q}{k_T}\boldsymbol{\Gamma}_0\right)\exp\left(-\frac{2\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\boldsymbol{\Gamma}_1\right)\exp\left(-\frac{2\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\boldsymbol{\Gamma}_1\right)\exp\left(-\frac{2\alpha_s}{\pi}\ln\frac{Q}{k_T}\boldsymbol{\Gamma}_0\right)$$

$$= \quad \exp\left(-\frac{4\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\mathbf{\Gamma}_1\right)\exp\left(-\frac{4\alpha_s}{\pi}\ln\frac{Q}{k_T}\mathbf{\Gamma}_0\right)$$

where

$$\mathbf{\Gamma}_0 = \left(\frac{4}{3}Y + \frac{4}{3}\rho\left(Y, |\Delta y|\right)\right) \tag{3.6}$$

and

$$\begin{aligned} \mathbf{\Gamma}_1 &= \left(\frac{2}{3}Y + \frac{1}{3}\rho(Y;2|y_3|) + \frac{1}{3}\rho(Y;2|y_4|)\right. \\ &+ \left.\frac{3}{4}\rho(Y;2|y_5|) - \frac{3}{4}\lambda\left(Y;|y_3|+|y_5|,|\phi_3-\phi_5|\right)\right) \end{aligned}$$

supressing the full arguments for clarity.

expanding the exponents in powers of α_s to $O(\alpha_s)^2$ with $b = -\frac{4\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\mathbf{\Gamma}_1$ and $a = -\frac{4\alpha_s}{\pi}\ln\frac{Q}{k_T}\mathbf{\Gamma}_0$ then:

$$\begin{aligned} \exp(a+b) &= 1 + (a+b) + \frac{1}{2}(a+b)^2 \\ &= 1 + a + b + \frac{a^2+b^2}{2} + ab \end{aligned}$$

therefore

$$\Omega_R = -\tfrac{4}{3}\omega_{34} \quad \left(1 + a + b + \frac{a^2+b^2}{2} + ab\right)\boldsymbol{M}_0^{\dagger}\boldsymbol{M}_0$$

$$
\begin{aligned}
\Omega_V &= \exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk'_T}{k'_T}\boldsymbol{\Gamma}_0^\dagger\right) \\
&\quad \times\exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\boldsymbol{\Gamma}_0\right)\boldsymbol{\gamma}_0\exp\left(-\frac{2\alpha_s}{\pi}\int_{k_T}^{Q}\frac{dk'_T}{k'_T}\boldsymbol{\Gamma}_0\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0+\text{c.c.} \\
&= 2\left(\frac{2}{3}\omega_{34}\exp\left(-\frac{2\alpha_s}{\pi}\ln\frac{Q}{Q_0}\boldsymbol{\Gamma}_0\right)\exp\left(-\frac{2\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\boldsymbol{\Gamma}_0\right)\right. \\
&\quad \left.\times\exp\left(-\frac{2\alpha_s}{\pi}\ln\frac{Q}{k_T}\boldsymbol{\Gamma}_0\right)\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0
\end{aligned}
$$

performing the k'_T integral, recognising $\boldsymbol{\Gamma}_0^\dagger=\boldsymbol{\Gamma}_0$, putting in the explicit expression for $\boldsymbol{\gamma}_0$ and therefore recognising that $(\Omega_V)^\dagger=\Omega_V$.

If $e=\left(-\frac{2\alpha_s}{\pi}\ln\frac{Q}{Q_0}\boldsymbol{\Gamma}_0\right)$, $d=\left(-\frac{2\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\boldsymbol{\Gamma}_0\right)$ and $c=\left(-\frac{2\alpha_s}{\pi}\ln\frac{Q}{k_T}\boldsymbol{\Gamma}_0\right)$ then expanding the exponents to $O(x)^2$ as

$$
\exp(e+d+c) = 1+e+d+c+\frac{1}{2}e^2+\frac{1}{2}d^2+\frac{1}{2}c^2+ed+ec+dc
$$

Therefore

$$
\Omega_V=\frac{4}{3}\omega_{34}\left(1+e+d+c+\frac{1}{2}e^2+\frac{1}{2}d^2+\frac{1}{2}c^2+ed+ec+dc\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0 \qquad (3.7)
$$

Then

$$
\begin{aligned}
\sigma_1 &= -\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\int_{out}\frac{dyd\phi}{2\pi} \\
&\quad \boldsymbol{M}_0^\dagger\boldsymbol{M}_0\left(-\frac{4}{3}\omega_{34}\left(1+a+b+\frac{a^2+b^2}{2}+ab\right)\right. \\
&\quad \left.+\frac{4}{3}\omega_{34}\left(1+e+d+c+\frac{1}{2}e^2+\frac{1}{2}d^2+\frac{1}{2}c^2+ed+ec+dc\right)\right)
\end{aligned}
$$

Note that $y = y_5$ and that only b depends upon y. Also note that the integral over y is convergent as $y \to \infty$ since in that limit $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_0$, $b = 2d$ and $a = 2c$ and the integrand vanishes.

As a further check that the $\boldsymbol{D}_{0\mu}$ and $\boldsymbol{\gamma}_0$ matrices are correct, with the exponents set to zero, i.e. the undressed cross-section for one gluon outside the gap, we would have:

$$
\begin{aligned}
\sigma_1 &= -\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\int_{out}\frac{dyd\phi}{2\pi}\boldsymbol{M}_0^\dagger\boldsymbol{M}_0\left(-\frac{4}{3}\omega_{34}\left(1\right)+\frac{4}{3}\omega_{34}\left(1\right)\right) \\
&= 0
\end{aligned}
$$

which is a statement of the Block-Nordsieck theorem (see Figure 1.1).

3.3.3 Two Gluons Outside The Gap

The cross-section for two gluons outside the gap is:

$$\sigma_2 = \left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_{1T}}\frac{dk_{2T}}{k_{2T}}\int_{out}\frac{dy_2 d\phi_2}{2\pi}\right)\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_{1T}}{k_{1T}}\int_{out}\frac{dy_1 d\phi_1}{2\pi}\right)$$
$$\times\left(\Omega_R+\Omega_V+\Omega_{RV}\right)$$

where the k_{1T} and k_{2T} integrals refer to the first and second out of gap gluons respectively and $k_{1T} \gg k_{2T}$. This calculation is being carried out to $O\left(\alpha_s^2\right)$ and thus there is no in gap soft gluon dressing. For this case there is a more complex structure to the Ω functions, with both a purely real, purely virtual and a mixed real-virtual component (in which the real emission maybe the first or second emission). This corresponds to the two gluon diagram cut in all possible ways. The emission matrix combinations needed are tabulated below:

Table 3.1: Emission Matrix Combinations

$\boldsymbol{M}_1$	$\boldsymbol{D}_{1\nu}(k_2)\boldsymbol{D}_{0\mu}(k_1)\lvert\boldsymbol{M}_0\rangle$
$\boldsymbol{M}_2$	$\boldsymbol{\gamma}_0(k_2)\boldsymbol{\gamma}_0(k_1)\lvert\boldsymbol{M}_0\rangle$
$\boldsymbol{M}_3$	$\boldsymbol{\gamma}_1(k_2)\boldsymbol{D}_{0\mu}(k_1)\lvert\boldsymbol{M}_0\rangle$
$\boldsymbol{M}_4$	$\boldsymbol{D}_{0\mu}(k_2)\boldsymbol{\gamma}_0(k_1)\lvert\boldsymbol{M}_0\rangle$
$\boldsymbol{M}_5$	$\boldsymbol{\gamma}_0(k_1)\lvert\boldsymbol{M}_0\rangle$
$\boldsymbol{M}_6$	$\boldsymbol{D}_{0\mu}(k_1)\lvert\boldsymbol{M}_0\rangle$
$\boldsymbol{M}_0$	$1\lvert\boldsymbol{M}_0\rangle$

The required inner products are therefore as tabulated below:

Calculation of the inner products is as follows:

$\boldsymbol{M}_1^{\dagger}\boldsymbol{M}_1$

Table 3.2: Two Gluon Outside The Gap Amplitudes

	$\boldsymbol{M}_1$	$\boldsymbol{M}_2$	$\boldsymbol{M}_3$	$\boldsymbol{M}_4$	$\boldsymbol{M}_5$	$\boldsymbol{M}_6$	$\boldsymbol{M}_0$
$\boldsymbol{M}_1^\dagger$	$\boldsymbol{M}_1^\dagger\boldsymbol{M}_1$	+	+	+	+	+	+
$\boldsymbol{M}_2^\dagger$	+	+	+	+	+	+	$\boldsymbol{M}_2^\dagger\boldsymbol{M}_0$
$\boldsymbol{M}_3^\dagger$	+	+	+	+	+	$\boldsymbol{M}_3^\dagger\boldsymbol{M}_6$	+
$\boldsymbol{M}_4^\dagger$	+	+	+	+	+	$\boldsymbol{M}_4^\dagger\boldsymbol{M}_6$	+
$\boldsymbol{M}_5^\dagger$	+	+	+	+	$\boldsymbol{M}_5^\dagger\boldsymbol{M}_5$	+	+
$\boldsymbol{M}_6^\dagger$	+	+	$\boldsymbol{M}_6^\dagger\boldsymbol{M}_3$	$\boldsymbol{M}_6^\dagger\boldsymbol{M}_4$	+	+	+
$\boldsymbol{M}_0^\dagger$	+	$\boldsymbol{M}_0^\dagger\boldsymbol{M}_2$	+	+	+	+	+

$$\boldsymbol{M}_1^\dagger\boldsymbol{M}_1 = \left(\boldsymbol{D}_{1\nu}(k_2)\,\boldsymbol{D}_{0\mu}(k_1)\right)^\dagger \left(\boldsymbol{D}_{1\nu}(k_2)\,\boldsymbol{D}_{0\mu}(k_1)\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0$$

For clarity the transverse momentum dependence is omitted until the Lorentz index is contracted.

Now $\left(\boldsymbol{D}_{1\nu}(k_2)\,\boldsymbol{D}_{0\mu}(k_1)\right)^\dagger \left(\boldsymbol{D}_{1\nu}(k_2)\,\boldsymbol{D}_{0\mu}(k_1)\right) =$

$$\frac{2}{\sqrt{3}}\left(h_{4\mu}-h_{3\mu}\right)^\dagger \begin{pmatrix} \frac{1}{\sqrt{6}}h_{4\nu}-\frac{1}{\sqrt{6}}h_{3\nu} \\ \sqrt{\frac{5}{12}}h_{4\nu}-\sqrt{\frac{5}{12}}h_{3\nu} \\ \frac{3}{\sqrt{12}}h_{4\nu}+\frac{3}{\sqrt{12}}h_{3\nu}-\frac{\sqrt{12}}{2}h_{5\nu} \end{pmatrix}^\dagger$$
$$\times \begin{pmatrix} \frac{1}{\sqrt{6}}h_{4\nu}-\frac{1}{\sqrt{6}}h_{3\nu} \\ \sqrt{\frac{5}{12}}h_{4\nu}-\sqrt{\frac{5}{12}}h_{3\nu} \\ \frac{3}{\sqrt{12}}h_{4\nu}+\frac{3}{\sqrt{12}}h_{3\nu}-\frac{\sqrt{12}}{2}h_{5\nu} \end{pmatrix}\frac{2}{\sqrt{3}}\left(h_{4\mu}-h_{3\mu}\right)$$

$$= \tfrac{4}{3}\left(h_{4\mu}-h_{3\mu}\right)^2\left(\left(\tfrac{1}{\sqrt{6}}h_{4\nu}-\tfrac{1}{\sqrt{6}}h_{3\nu}\right)^2+\left(\sqrt{\tfrac{5}{12}}h_{4\nu}-\sqrt{\tfrac{5}{12}}h_{3\nu}\right)^2\right.$$

$$\left.+\left(\frac{3}{\sqrt{12}}h_{4\nu}+\frac{3}{\sqrt{12}}h_{3\nu}-\frac{\sqrt{12}}{2}h_{5\nu}\right)\left(\frac{3}{\sqrt{12}}h_{4\nu}+\frac{3}{\sqrt{12}}h_{3\nu}-\frac{\sqrt{12}}{2}h_{5\nu}\right)\right)$$

$$\begin{aligned}
&= \frac{4}{3}\left(-2h_{3\mu}h_{4\mu}\right)\left(-\frac{1}{3}h_{3\nu}h_{4\nu}-\frac{5}{6}h_{3\nu}h_{4\nu}+\frac{3}{2}h_{3\nu}h_{4\nu}-3\left(h_{3\nu}h_{5\nu}+h_{4\nu}h_{5\nu}\right)\right)\\
&= -\frac{8}{3}h_{3\mu}h_{4\mu}\left(\frac{1}{3}h_{3\nu}h_{4\nu}-3\left(h_{3\nu}h_{5\nu}+h_{4\nu}h_{5\nu}\right)\right)\\
&= -\frac{4}{3}\omega_{34}(k_1)\left(\frac{1}{6}\omega_{34}(k_2)-\frac{3}{2}\left(\omega_{35}(k_2)+\omega_{45}(k_2)\right)\right)\\
&= -\frac{2}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)+2\omega_{34}(k_1)\left(\omega_{35}(k_2)+\omega_{45}(k_2)\right)
\end{aligned}$$

Therefore

$$\boldsymbol{M}_1^\dagger\boldsymbol{M}_1 = \left(-\frac{2}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)+2\omega_{34}(k_1)\left(\omega_{35}(k_2)+\omega_{45}(k_2)\right)\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0$$

$\boldsymbol{M}_0^\dagger\boldsymbol{M}_2$

$$\boldsymbol{M}_0^\dagger\boldsymbol{M}_2 = (1)^\dagger\left(\boldsymbol{\gamma}_0(k_2)\,\boldsymbol{\gamma}_0(k_1)\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0$$

as

$$\boldsymbol{\gamma}_0 = \tfrac{2}{3}\omega_{34}$$

then

$$\boldsymbol{M}_0^\dagger \boldsymbol{M}_2 \quad = \frac{4}{9}\left(\omega_{34}\left(k_1\right)\omega_{34}\left(k_2\right)\right)\boldsymbol{M}_0^\dagger \boldsymbol{M}_0$$

and it can clearly be seen that:

$$\boldsymbol{M}_0^\dagger \boldsymbol{M}_2 = \boldsymbol{M}_2^\dagger \boldsymbol{M}_0 \tag{3.8}$$

$\boldsymbol{M}_6^\dagger \boldsymbol{M}_3$

$$\boldsymbol{M}_6^\dagger \boldsymbol{M}_3 \quad = \left(\boldsymbol{D}_{0\mu}\left(k_1\right)\right)^\dagger \left(\boldsymbol{\gamma}_1\left(k_2\right)\boldsymbol{D}_{0\mu}\left(k_1\right)\right)\boldsymbol{M}_0^\dagger \boldsymbol{M}_0$$

now

$$\begin{aligned}\left(\boldsymbol{D}_{0\mu}\left(k_1\right)\right)^\dagger \left(\boldsymbol{\gamma}_1\left(k_2\right)\boldsymbol{D}_{0\mu}\left(k_1\right)\right) \quad &= \quad \frac{2}{\sqrt{3}}\left(h_{3\mu}-h_{4\mu}\right)^\dagger \\ &\quad \times\left(\frac{3}{4}\left(\omega_{35}+\omega_{45}\right)-\frac{1}{12}\omega_{34}\right)\frac{2}{\sqrt{3}}\left(h_{3\mu}-h_{4\mu}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{3}\left(-\omega_{34}\left(k_1\right)\right)\left(\frac{3}{4}\left(\omega_{35}\left(k_2\right)+\omega_{45}\left(k_2\right)\right)-\frac{1}{12}\omega_{34}\left(k_2\right)\right) \\
&= \frac{1}{9}\omega_{34}\left(k_1\right)\omega_{34}\left(k_2\right)-\omega_{34}\left(k_1\right)\left(\omega_{35}\left(k_2\right)+\omega_{45}\left(k_2\right)\right)
\end{aligned}
$$

Therefore

$$
\boldsymbol{M}_6^{\dagger}\boldsymbol{M}_3 = \left(\frac{1}{9}\omega_{34}\left(k_1\right)\omega_{34}\left(k_2\right)-\omega_{34}\left(k_1\right)\left(\omega_{35}\left(k_2\right)+\omega_{45}\left(k_2\right)\right)\right)\boldsymbol{M}_0^{\dagger}\boldsymbol{M}_0
$$

and it can clearly be seen that:

$$
\boldsymbol{M}_6^{\dagger}\boldsymbol{M}_3 = \boldsymbol{M}_3^{\dagger}\boldsymbol{M}_6 \tag{3.9}
$$

$\boldsymbol{M}_6^{\dagger}\boldsymbol{M}_4$

$$
\boldsymbol{M}_6^{\dagger}\boldsymbol{M}_4 = \left(\boldsymbol{D}_{0\mu}\left(k_2\right)\right)^{\dagger}\left(\boldsymbol{D}_{0\mu}\left(k_2\right)\boldsymbol{\gamma}_0\left(k_1\right)\right)\boldsymbol{M}_0^{\dagger}\boldsymbol{M}_0
$$

The $\left(\boldsymbol{D}_{0\mu}\right)^{\dagger}$operator has k_2 dependence here to conserve momentum across the cut.

Now

$$
\begin{aligned}
\left(\boldsymbol{D}_{0\mu}\left(k_2\right)\right)^{\dagger}\boldsymbol{D}_{0\mu}\left(k_2\right)\boldsymbol{\gamma}_0\left(k_1\right) &= \frac{2}{\sqrt{3}}\left(h_{3\mu}-h_{4\mu}\right)^{\dagger}\frac{2}{\sqrt{3}}\left(h_{3\mu}-h_{4\mu}\right)\frac{2}{3}\omega_{34}\left(k_1\right) \\
&= -\frac{8}{9}\omega_{34}\left(k_2\right)\omega_{34}\left(k_1\right)
\end{aligned}
$$

therefore

$$\boldsymbol{M}_6^\dagger \boldsymbol{M}_4 \quad = -\frac{8}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)\,\boldsymbol{M}_0^\dagger \boldsymbol{M}_0$$

and it can clearly be seen that:

$$\boldsymbol{M}_6^\dagger \boldsymbol{M}_4 = \boldsymbol{M}_4^\dagger \boldsymbol{M}_6 \tag{3.10}$$

$\boldsymbol{M}_5^\dagger \boldsymbol{M}_5$

$$\boldsymbol{M}_5^\dagger \boldsymbol{M}_5 \quad = (\boldsymbol{\gamma}_0(k_1))^\dagger (\boldsymbol{\gamma}_0(k_2))\,\boldsymbol{M}_0^\dagger \boldsymbol{M}$$

now

$$\begin{aligned}(\boldsymbol{\gamma}_0(k_1))^\dagger (\boldsymbol{\gamma}_0(k_2)) \quad &= \quad \left(\frac{2}{3}\omega_{34}(k_1)\right)^\dagger \left(\frac{2}{3}\omega_{34}(k_2)\right) \\ &= \quad \frac{4}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)\end{aligned}$$

where k_2 is used after the re-expression of the double transverse momentum integral (see below).

Therefore

$$\boldsymbol{M}_5^\dagger \boldsymbol{M}_5 \quad = \frac{4}{9}\omega_{34}\left(k_1\right)\omega_{34}\left(k_2\right)\boldsymbol{M}_0^\dagger \boldsymbol{M}_0$$

The double integral $\left(\int_{Q_0}^{k_{1T}} \frac{dk_{2T}}{k_{2T}}\right)\left(\int_{Q_0}^{Q} \frac{dk_{1T}}{k_{1T}}\right)$ may be re-expressed as

$$\frac{1}{2}\left(\int_{Q_0}^{Q} \frac{dk_{2T}}{k_{2T}}\right)\left(\int_{Q_0}^{Q} \frac{dk_{1T}}{k_{1T}}\right).$$

With the range of intergration for both k_{1T} and k_{2T} being now over Q to Q_0. This introduces a factor of $\frac{1}{2}$ before $\boldsymbol{M}_1^\dagger \boldsymbol{M}_1; \boldsymbol{M}_2^\dagger \boldsymbol{M}_0, \boldsymbol{M}_0^\dagger \boldsymbol{M}_2; \boldsymbol{M}_3^\dagger \boldsymbol{M}_6, \boldsymbol{M}_6^\dagger \boldsymbol{M}_3$ and $\boldsymbol{M}_4^\dagger \boldsymbol{M}_6, \boldsymbol{M}_6^\dagger \boldsymbol{M}_4$ in evaluation of the cross-section. Given that $\boldsymbol{M}_2^\dagger \boldsymbol{M}_0, \boldsymbol{M}_3^\dagger \boldsymbol{M}_6$ and $\boldsymbol{M}_4^\dagger \boldsymbol{M}_6$ are equal to their Hermitian conjugates then the factor of $\frac{1}{2}$ is mitigated by the pairing. As the argument of both $\boldsymbol{\gamma}_0$'s runs over the full range of Q to Q_0 there is no factor of a $\frac{1}{2}$ before the $\boldsymbol{M}_5^\dagger \boldsymbol{M}_5$ expression.

Therefore

$$
\begin{aligned}
(\Omega_R+\Omega_V+\Omega_{RV}) &= \frac{1}{2}\boldsymbol{M}_1^\dagger\boldsymbol{M}_1+\boldsymbol{M}_2^\dagger\boldsymbol{M}_0+\boldsymbol{M}_5^\dagger\boldsymbol{M}_5+\boldsymbol{M}_3^\dagger\boldsymbol{M}_6+\boldsymbol{M}_4^\dagger\boldsymbol{M}_6 \\
&= \left(\frac{1}{2}\left(-\frac{2}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)+2\omega_{34}(k_1)\,(\omega_{35}(k_2)+\omega_{45}(k_2))\right)\right. \\
&+ \frac{4}{9}(\omega_{34}(k_1)\,\omega_{34}(k_2))+\frac{4}{9}\omega_{34}(k_1)\,\omega_{34}(k_2) \\
&+ \frac{1}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)-\omega_{34}(k_1)\,(\omega_{35}(k_2)+\omega_{45}(k_2)) \\
&- \left.\frac{8}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0 \\
&= \left(-\frac{1}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)+\omega_{34}(k_1)\,(\omega_{35}(k_2)+\omega_{45}(k_2))\right. \\
&+ \frac{1}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)-\omega_{34}(k_1)\,(\omega_{35}(k_2)+\omega_{45}(k_2)) \\
&+ \left.\frac{8}{9}(\omega_{34}(k_1)\,\omega_{34}(k_2))-\frac{8}{9}\omega_{34}(k_1)\,\omega_{34}(k_2)\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0 \\
&= 0
\end{aligned}
$$

Therefore to $O(\alpha_s)^2$:

$$
\sigma_2 = 0
$$

As for the undressed cross-section for one gluon outside the gap this cancellation is a further manifestation of the Bloch-Nordsieck Theorem and again provides a check of the emission matrices.

Chapter 4

Conclusion

The gaps between jets paradigm, with a cut-off in transverse momentum for radiation within the gap, has lead to the discovery of both global and non-global soft gluon corrections to the basic scattering amplitude. The use of the eikonal approximation appropriate for soft gluons leads to logarithmic corrections of the ratio of the hard to the veto scale. The complexity of the colour albebra has meant that global and non-global leading logarithmic corrections to all orders have only been performed in the large N_c limit using the BSM evolution equation.

An alternative method of resummation of the leading logarithms is to calculate the corrections coming from a small number of gluons outside of the gap, dressed to an arbitrary order with in gap virtual gluons whilst keeping the full $N_c = 3$ dependence.

This thesis has involved the calculation of the in-gap virtual dressing matrices for zero, one and two out of gap gluons and the emission matrices for one and two real and eikonal out of gap gluons. The calculation of the cross-sections, as corrections to the Born cross-section, for zero gluons outside the gap to $O(\alpha_s)^3$, one gluon outside the gap to the same order and two gluons outside the gap to $O(\alpha_s)^2$ has then been performed.

The cross-section for zero gluons outside the gap to $O(\alpha_s)^3$ is:

$$\left(1+2a+2a^2+\frac{4}{3}a^3\right)\boldsymbol{M}_0^\dagger\boldsymbol{M}_0 \tag{4.1}$$

where $a = -\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\Gamma_0$.

For one gluon outside the gap the cross-section to $O(\alpha_s)^3$ is:

$$\begin{aligned}\sigma_1 &= -\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\int_{out}\frac{dyd\phi}{2\pi}\\ &\quad \boldsymbol{M}_0^\dagger\boldsymbol{M}_0\left(-\frac{4}{3}\omega_{34}\left(1+a+b+\frac{a^2+b^2}{2}+ab\right)\right.\\ &\quad \left.+\frac{4}{3}\omega_{34}\left(1+e+d+c+\frac{1}{2}e^2+\frac{1}{2}d^2+\frac{1}{2}c^2+ed+ec+dc\right)\right)\end{aligned}$$

where $a = -\frac{4\alpha_s}{\pi}\ln\frac{Q}{k_T}\boldsymbol{\Gamma}_0$, $b = -\frac{4\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\boldsymbol{\Gamma}_1$, $c = a/2$, $d = \left(-\frac{2\alpha_s}{\pi}\ln\frac{k_T}{Q_0}\boldsymbol{\Gamma}_0\right)$ and $e = \left(-\frac{2\alpha_s}{\pi}\ln\frac{Q}{Q_0}\boldsymbol{\Gamma}_0\right)$.

For two gluons outside the gap the cross-section to $O(\alpha_s)^2$ is zero.

Due to time constraints the expansion to $O(\alpha_s)^3$ for two gluons outside the gap could not be included in this thesis. However the Anomalous Dimension Matrix for the scenario of two gluons on the left has been calculated and is:

$$\begin{pmatrix} \begin{array}{c}\frac{2}{3}Y+\frac{1}{3}\rho_3+\frac{1}{3}\rho_4\\ +\frac{3}{4}\rho_5+\frac{3}{4}\rho_6\\ -\frac{3}{2}\lambda_{56}\end{array} & 0 & \frac{1}{2\sqrt{2}}\lambda_{35}-\frac{1}{2\sqrt{2}}\lambda_{36} \\ 0 & \begin{array}{c}\frac{2}{3}Y+\frac{1}{3}\rho_3+\frac{1}{3}\rho_4\\ +\frac{3}{4}\rho_5+\frac{3}{4}\rho_6-\frac{3}{8}\lambda_{35}\\ -\frac{3}{8}\lambda_{36}-\frac{3}{4}\lambda_{56}\end{array} & \frac{\sqrt{5}}{8}\lambda_{35}-\frac{\sqrt{5}}{8}\lambda_{36} \\ \frac{1}{2\sqrt{2}}\lambda_{35}-\frac{1}{2\sqrt{2}}\lambda_{36} & \frac{\sqrt{5}}{8}\lambda_{35}-\frac{\sqrt{5}}{8}\lambda_{36} & \begin{array}{c}\frac{2}{3}Y+\frac{1}{3}\rho_3+\frac{1}{3}\rho_4\\ +\frac{3}{4}\rho_5+\frac{3}{4}\rho_6\\ -\frac{3}{8}\lambda_{35}-\frac{3}{8}\lambda_{36}-\frac{3}{4}\lambda_{56}\end{array} \end{pmatrix} \tag{4.2}$$

The final calculation needed to $O(\alpha_s)^3$ is the undressed cross-section for three gluons outside the gap.

The undressed cross-section for both one and two gluons outside the gap is seen to be zero as predicted by the Bloch-Nordsieck Theorem and provides a useful check of the real and virtual emission matrices.

When completed to $O(\alpha_s)^3$ for zero, one, two and three gluons outside the gap and expressed in the general N_c form, the cross-sectional corrections will provide an interesting comparison with the Banfi-Marchesini-Smye equation predictions carried out for large N_c.

A better understanding of gluon radiation is particularly pertinent at the current time. The search for the Higgs boson at the Large Hadron Collider includes looking for Higgs production in association with two jets in a similar scenario to that of “gaps between jets”. Higgs production in this setting can occur via gluon-gluon fusion and weak boson fusion. Gluon radiation is different in the two cases and thus applying a veto on this radiation between the jets allows the weak boson channel to be enhanced [11]. A clearer understanding of gluon radiation is thus important in studying the coupling of the Higgs boson to the weak bosons in this channel [12, 13].

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Erratum

1. page 13; para 3; line 1. After "where $\alpha_s$is the strong coupling constant" add "which we take to be fixed throughout this thesis".
2. page 13; equation 1.1. Change to read $\sigma = \sigma_0 \sum_{n=0}^{\infty} c_n \alpha_s^n \ln^n \left(\frac{Q}{Q_0}\right)$ where $c_n$are the logarithmic coefficients.
3. page 81; equation 3.6. Change coefficients of Y and ρ from $\frac{4}{3}$ to $\frac{2}{3}$.
4. page 76; line 12. Discard equation:

$$-\frac{2}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\alpha_s\left(\frac{2}{3}Y+\frac{2}{3}\rho\left(Y,|\Delta y|\right)\right) \quad = \quad -\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\left(\frac{4}{3}Y+\frac{4}{3}\rho\left(Y,|\Delta y|\right)\right)$$

5. page 76; line 14. In $\mathbf{\Gamma_0} = \left(\frac{4}{3}Y + \frac{4}{3}\rho\left(Y,|\Delta y|\right)\right)$ change coefficients of Y and ρ from $\frac{4}{3}$ to $\frac{2}{3}$.
6. page 77; line 1. In $\mathbf{M} = \left(1 - \left(-\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\mathbf{\Gamma_0}\right) + \frac{1}{2}\left(-\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\mathbf{\Gamma_0}\right)^2 - \frac{1}{6}\left(-\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\mathbf{\Gamma_0}\right)^3\right)\mathbf{M}_0$ insert 2 before each α_s.
7. page 77; line 3. In $a = -\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\mathbf{\Gamma_0}$ insert 2 before α_s.
8. page 93; line 6. In $a = -\frac{\alpha_s}{\pi}\ln\frac{Q}{Q_0}\mathbf{\Gamma_0}$ insert 2 before α_s.