PC3642: ELECTRODYNAMICS

Apostolos Pilaftsis

Contents

PC3642 Electrodynamics

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BOOKS

→⋆ Heald & Marion *Classical em radiation* 3rd edition (1995) Saunders College Pub. (right level, covers course but Gaussian units, Blackwells, Departmental library).

Barger & Olsson *Classical electricity and magnetism* (1987) Allyn & Bacon (out of print, SI units, 538.3/B17).

Vanderlinde *Classical em theory* (1993) Wiley (SI units, covers most of course, uses metric tensor, 538.3/V1)

Feynman *The Feynman lectures on physics* Vol II (1964) Addison-Wesley. (SI units, 530.4)

Alan M. Portis *Electromagnetic fields, sources and media* (1978) Wiley (SI units, 538.3/P26).

∗ Jackson *Classical electrodynamics* (1998) Wiley (Third edition has SI units for everything except relativistic electrodynamics (Gaussian), advanced text, covers everything, 538.3/J4)

SYLLABUS

1. **Linear Algebra** Revision of vectors and matrices; basis sets and components. Index notation and summation convention. Rotational invariance and cartesian tensors.

2. **Electromagnetic Field Equations** Maxwell's equations and wave solutions. Definition of scalar and vector potential. Poisson's equation and electro- and magnetostatics; multipole expansions. Electrodynamics in Lorentz Gauge.

3. **Harmonic Sources and Radiation** Multipole radiation: electric (Hertzian) and magnetic dipole radiation; slow-down of pulsars. Interaction of radiation with matter: Rayleigh and Thompson scattering; propagation through plasmas.

4. **Accelerating Charges** Retarded potentials and fields. Lienard-Wiechert potentials; fields around a moving single charge. Larmor power formula; synchrotron radiation; brehmstrahlung.

5. **Electromagnetism and Relativity** Four vectors and tensors; relativistic dynamics. Consistency of Maxwell's equations and relativity. Electromagnetic field tensor and electrodynamics in covariant form.

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- Prerequisites in Mathematics

PC3642 Cartesian vectors and tensors

Vectors

Orthogonal unit vectors: $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ or $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ (but not **i**, **j**, **k**).

They form an orthonormal basis: $\mathbf{e}_i.\mathbf{e}_j{=}\delta_{ij}$; ($\delta_{ij} = 0$ if $i \neq j$; $\delta_{ij} = 1$ if $i = j$)

The following all represent the same vector:

$$\mathbf{x} \qquad x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \qquad (x_1, x_2, x_3) \qquad \sum_i x_i\mathbf{e}_i \qquad x_i\mathbf{e}_i$$

(note summation convention above: sum over repeated indicies is assumed)

Components of $\mathbf{x}$ are x_i where $x_i = \mathbf{x}.\mathbf{e}_i$

Length or magnitude: $x = +\sqrt{x_1^2 + x_2^2 + x_3^2}$

$\mathbf{e}_1{=}(1, 0, 0)$ $\mathbf{e}_2{=}(0, 1, 0)$ $\mathbf{e}_3{=}(0, 0, 1)$

Matrices

$$\mathbf{M}\,\mathbf{x} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \mathbf{y}$$

This is a matrix $\times$ column vector: elements of $\mathbf{y}$ are $y_i = \sum_j M_{ij}x_j = M_{ij}x_j$

The transpose of $\mathbf{M}{=}(M_{ij})$ is $\mathbf{M}^{\mathbf{T}}{=}(M_{ji})$

$$\text{if} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{then} \qquad \mathbf{x}^{\mathbf{T}} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} = (\,x^2\,) \qquad \mathbf{x}\,\mathbf{x}^{\mathbf{T}} = \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{pmatrix}$$

Rotations

If a set of rectangular axes $\mathbf{e}_i$ are rotated to a new orientation specified by $\mathbf{e}'_i$ then the components of a vector in the primed system is related to the components in the un-primed system by

$$\mathbf{x}' = \mathbf{R}\mathbf{x} \qquad \text{or} \qquad x'_i = \sum_j R_{ij}x_j$$

where the rotation matrix $\mathbf{R}$ has elements R_{ij} equal to the cosine of the angle between the $\mathbf{e}'_i$ axis and the $\mathbf{e}_j$ axis: $R_{ij} = \mathbf{e}'_i.\mathbf{e}_j$

The inverse of $\mathbf{R}$ obviously has elements $\mathbf{e}_i.\mathbf{e}'_j$ and so is simply the transpose of $\mathbf{R}$: $\mathbf{R}^{-1}$=$\mathbf{R}^{\mathbf{T}}$. Such a matrix (and transformation) is said to be orthogonal. Since it is only the *representation* of the vector that changes with choice of coordinate system (and not the vector itself) it is obvious that an orthogonal transfomation preserves lengths and relative orientations of vectors ($\mathbf{x}'.\mathbf{y}'$=$\mathbf{x}.\mathbf{y}$=an invariant scalar). For a rotation θ about the z-axis we have

$$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix which represents the operator $\mathbf{M}$=(M_{ij}) in the new basis is $\mathbf{M}'$=$\mathbf{R}$ $\mathbf{M}$ $\mathbf{R}^{\mathbf{T}}$. The matrix elements are therefore

$$M'_{ij} \quad = \quad \sum_k R_{ik}(\sum_l M_{kl} R^{\mathrm{T}}_{lj}) \quad = \quad \sum_{kl} R_{ik} R_{jl} M_{kl}$$

We can readily see the matrix transforms in this way by considering the transformation of the equation $\mathbf{y}$=$\mathbf{M}$ $\mathbf{x}$ where $\mathbf{y}$ and $\mathbf{x}$ are column vectors:

$$\mathbf{y}' = \mathbf{R}\ \mathbf{y} = \mathbf{R}\ (\mathbf{M}\ \mathbf{x}) = \mathbf{R}\ (\mathbf{M}\ \mathbf{R}^{\mathbf{T}}\mathbf{R}\ \mathbf{x}) = \mathbf{R}\ \mathbf{M}\ \mathbf{R}^{\mathbf{T}}(\mathbf{R}\ \mathbf{x}) = \mathbf{M}'\mathbf{x}'$$

Definition of a tensor

We may generalise the above to define tensors. In n-dimensional space a tensor $\mathbf{T}$ of rank m is a set of n^m quantities that transform under a coordinate rotation in the following way:

$$T'_{abcd...} = \sum_{ijkl...} R_{ai} R_{bj} R_{ck} R_{dl} ... T_{ijkl...}$$

Thus a tensor of rank 0 is a scalar:

$$a' = a.$$

A tensor of rank 1 is a vector:

$$x'_i = \sum_j R_{ij} x_j.$$

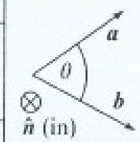
A tensor of rank 2 in 3-dimensional space is a 3×3 matrix:

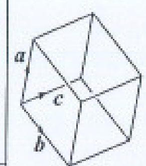
$$M'_{ij} = \sum_{kl} R_{ik} R_{jl} M_{kl}.$$

2.2 Vectors and matrices

Vector algebra

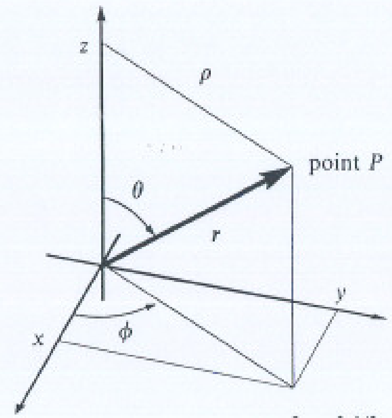
Scalar product[a]	$a\cdot b=\lvert a\rvert\lvert b\rvert\cos\theta$	(2.1)
Vector product[b]	$a\times b=\lvert a\rvert\lvert b\rvert\sin\theta\,\hat{n}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$	(2.2)
Product rules	$a\cdot b=b\cdot a$	(2.3)
	$a\times b=-b\times a$	(2.4)
	$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$	(2.5)
	$a\times(b+c)=(a\times b)+(a\times c)$	(2.6)
Lagrange's identity	$(a\times b)\cdot(c\times d)=(a\cdot c)(b\cdot d)-(a\cdot d)(b\cdot c)$	(2.7)
Scalar triple product	$(a\times b)\cdot c=\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}$	(2.8)
	$=(b\times c)\cdot a=(c\times a)\cdot b$	(2.9)
	$=$ volume of parallelepiped	(2.10)
Vector triple product	$(a\times b)\times c=(a\cdot c)b-(b\cdot c)a$	(2.11)
	$a\times(b\times c)=(a\cdot c)b-(a\cdot b)c$	(2.12)
Reciprocal vectors	$a'=(b\times c)/[(a\times b)\cdot c]$	(2.13)
	$b'=(c\times a)/[(a\times b)\cdot c]$	(2.14)
	$c'=(a\times b)/[(a\times b)\cdot c]$	(2.15)
	$(a'\cdot a)=(b'\cdot b)=(c'\cdot c)=1$	(2.16)
Vector a with respect to a nonorthogonal basis $\{e_1,e_2,e_3\}$[c]	$a=(e_1'\cdot a)e_1+(e_2'\cdot a)e_2+(e_3'\cdot a)e_3$	(2.17)





[a] Also known as the "dot product" or the "inner product."
[b] Also known as the "cross-product." $\hat{n}$ is a unit vector making a right-handed set with a and b.
[c] The prime (′) denotes a reciprocal vector.

Common three-dimensional coordinate systems



$x=\rho\cos\phi=r\sin\theta\cos\phi$	(2.18)	$\rho=(x^2+y^2)^{1/2}$	(2.21)
$y=\rho\sin\phi=r\sin\theta\sin\phi$	(2.19)	$r=(x^2+y^2+z^2)^{1/2}$	(2.22)
$z=r\cos\theta$	(2.20)	$\theta=\arccos(z/r)$	(2.23)
		$\phi=\arctan(y/x)$	(2.24)

coordinate system:	rectangular	spherical polar	cylindrical polar
coordinates of P:	(x,y,z)	(r,θ,ϕ)	(ρ,ϕ,z)
volume element:	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$	$r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$	$\rho\,\mathrm{d}\rho\,\mathrm{d}z\,\mathrm{d}\phi$
metric elements[a] (h_1,h_2,h_3):	$(1,1,1)$	$(1,r,r\sin\theta)$	$(1,\rho,1)$

[a] In an orthogonal coordinate system (parameterised by coordinates q_1,q_2,q_3), the differential line element $\mathrm{d}l$ is obtained from $(\mathrm{d}l)^2=(h_1\,\mathrm{d}q_1)^2+(h_2\,\mathrm{d}q_2)^2+(h_3\,\mathrm{d}q_3)^2$.

Gradient

Rectangular coordinates	$\nabla f=\frac{\partial f}{\partial x}\hat{x}+\frac{\partial f}{\partial y}\hat{y}+\frac{\partial f}{\partial z}\hat{z}$	(2.25)	f scalar field; $\hat{}$ unit vector
Cylindrical coordinates	$\nabla f=\frac{\partial f}{\partial \rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi}+\frac{\partial f}{\partial z}\hat{z}$	(2.26)	ρ distance from the z axis
Spherical polar coordinates	$\nabla f=\frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$	(2.27)	
General orthogonal coordinates	$\nabla f=\frac{\hat{q}_1}{h_1}\frac{\partial f}{\partial q_1}+\frac{\hat{q}_2}{h_2}\frac{\partial f}{\partial q_2}+\frac{\hat{q}_3}{h_3}\frac{\partial f}{\partial q_3}$	(2.28)	q_i basis; h_i metric elements

Divergence

Rectangular coordinates	$\nabla\cdot A=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	(2.29)	A vector field; A_i ith component of A
Cylindrical coordinates	$\nabla\cdot A=\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$	(2.30)	ρ distance from the z axis
Spherical polar coordinates	$\nabla\cdot A=\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$	(2.31)	
General orthogonal coordinates	$\nabla\cdot A=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3)+\frac{\partial}{\partial q_2}(A_2h_3h_1)+\frac{\partial}{\partial q_3}(A_3h_1h_2)\right]$	(2.32)	q_i basis; h_i metric elements

Curl

Rectangular coordinates	$\nabla\times A=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{vmatrix}$	(2.33)	$\hat{}$ unit vector; A vector field; A_i ith component of A
Cylindrical coordinates	$\nabla\times A=\begin{vmatrix}\hat{\rho}/\rho & \hat{\phi} & \hat{z}/\rho\\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$	(2.34)	ρ distance from the z axis
Spherical polar coordinates	$\nabla\times A=\begin{vmatrix}\hat{r}/(r^2\sin\theta) & \hat{\theta}/(r\sin\theta) & \hat{\phi}/r\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ A_r & rA_\theta & rA_\phi\sin\theta\end{vmatrix}$	(2.35)	
General orthogonal coordinates	$\nabla\times A=\frac{1}{h_1h_2h_3}\begin{vmatrix}\hat{q}_1h_1 & \hat{q}_2h_2 & \hat{q}_3h_3\\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}$	(2.36)	q_i basis; h_i metric elements

Radial forms[a]

$\nabla r=\frac{r}{r}$	(2.37)	$\nabla(1/r)=\frac{-r}{r^3}$	(2.41)
$\nabla\cdot r=3$	(2.38)	$\nabla\cdot(r/r^2)=\frac{1}{r^2}$	(2.42)
$\nabla r^2=2r$	(2.39)	$\nabla(1/r^2)=\frac{-2r}{r^4}$	(2.43)
$\nabla\cdot(rr)=4r$	(2.40)	$\nabla\cdot(r/r^3)=4\pi\delta(r)$	(2.44)

[a]Note that the curl of any purely radial function is zero. $\delta(r)$ is the Dirac delta function.

Laplacian (scalar)

Rectangular coordinates	$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$	(2.45)	f scalar field
Cylindrical coordinates	$\nabla^2 f=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}$	(2.46)	ρ distance from the z axis
Spherical polar coordinates	$\nabla^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$	(2.47)	
General orthogonal coordinates	$\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$	(2.48)	q_i basis; h_i metric elements

Differential operator identities

$\nabla(fg)\equiv f\nabla g+g\nabla f$	(2.49)	
$\nabla\cdot(fA)\equiv f\nabla\cdot A+A\cdot\nabla f$	(2.50)	
$\nabla\times(fA)\equiv f\nabla\times A+(\nabla f)\times A$	(2.51)	
$\nabla(A\cdot B)\equiv A\times(\nabla\times B)+(A\cdot\nabla)B+B\times(\nabla\times A)+(B\cdot\nabla)A$	(2.52)	
$\nabla\cdot(A\times B)\equiv B\cdot(\nabla\times A)-A\cdot(\nabla\times B)$	(2.53)	f,g scalar fields
$\nabla\times(A\times B)\equiv A(\nabla\cdot B)-B(\nabla\cdot A)+(B\cdot\nabla)A-(A\cdot\nabla)B$	(2.54)	A,B vector fields
$\nabla\cdot(\nabla f)\equiv\nabla^2 f\equiv\triangle f$	(2.55)	
$\nabla\times(\nabla f)\equiv\mathbf{0}$	(2.56)	
$\nabla\cdot(\nabla\times A)\equiv 0$	(2.57)	
$\nabla\times(\nabla\times A)\equiv\nabla(\nabla\cdot A)-\nabla^2 A$	(2.58)	

Vector integral transformations

Gauss's (Divergence) theorem	$\int_V(\nabla\cdot A)\,\mathrm{d}V=\oint_{S_c}A\cdot\mathrm{d}s$	(2.59)	A vector field; $\mathrm{d}V$ volume element; S_c closed surface; V volume enclosed; S surface; $\mathrm{d}s$ surface element
Stokes's theorem	$\int_S(\nabla\times A)\cdot\mathrm{d}s=\oint_L A\cdot\mathrm{d}l$	(2.60)	L loop bounding S; $\mathrm{d}l$ line element
Green's first theorem	$\oint_S(f\nabla g)\cdot\mathrm{d}s=\int_V\nabla\cdot(f\nabla g)\,\mathrm{d}V$	(2.61)	f,g scalar fields
	$=\int_V[f\nabla^2 g+(\nabla f)\cdot(\nabla g)]\,\mathrm{d}V$	(2.62)	
Green's second theorem	$\oint_S[f(\nabla g)-g(\nabla f)]\cdot\mathrm{d}s=\int_V(f\nabla^2 g-g\nabla^2 f)\,\mathrm{d}V$	(2.63)	

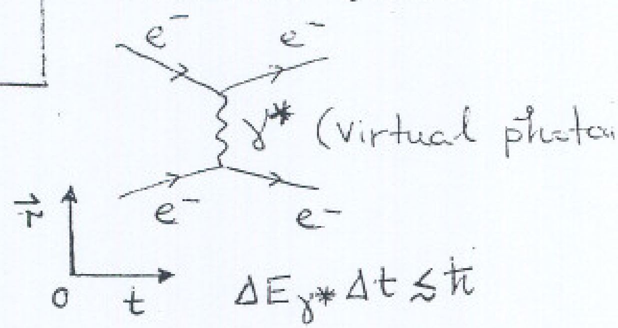
- ## Introduction – Motivation

Electrodynamics (ED)

↓

Quantum Electrodynamics (QED)

Feynman graph:



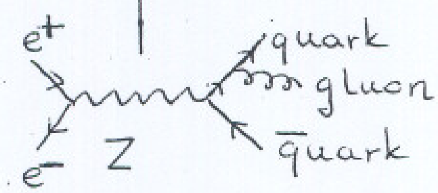
$\Delta E_{\gamma^*} \Delta t \lesssim \hbar$

↓

Standard Model (SM)

QED	Weak	Strong Forces
γ photons	$W^\pm, Z$ bosons	g (gluons)

Theory tested at CERN



⇣

Quantization of Gravity ?

Our hopes $\longmapsto$ $\begin{cases} \text{Unified Theories, Supersymmetry} \\ \text{and Superstrings} \end{cases}$

Our focus is on the Theory of Classical ED:

Limit of QED for small Energy-Momentum Transfers

- Units

	SI	Gaussian
F_{el}	$=\frac{q_1 q_2}{4\pi\epsilon_0 r^2}$	$=\frac{q_1 q_2}{r^2}$
F_{mag}	$=\frac{\mu_o I_1 I_2}{2\pi r}$	$=\frac{2 I_1 I_2}{c^2 r}$
$\nabla.\mathbf{E}$	$=\rho/\epsilon_0$	$=4\pi\rho$
$\nabla\times\mathbf{E}$	$=-\frac{\partial\mathbf{B}}{\partial t}$	$=\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$
$\nabla\times\mathbf{B}$	$=\mu_0\mathbf{j}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$	$=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$
$\mathbf{F}$	$=q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$	$=q(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B})$

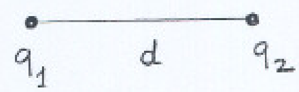
Gaussian units make Maxwell's equations appear more symmetric but hide the distinction between $\mathbf{B}/\mathbf{H}$ and $\mathbf{D}/\mathbf{E}$. The time t always occurs alongside c which is useful in relativity when the fourth component of 4-D space is ict. See the footnote in Heald and Marion, p.137 on general conversion.

I. Electromagnetic Field Equations

I.1 Maxwell's Equations

Considerations in free space (vacuum)

Coulomb's Law :

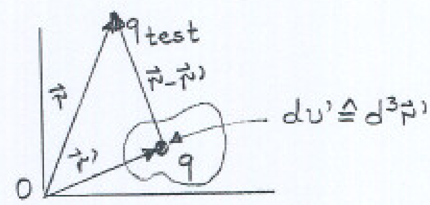


Force $|\vec{F}_{12}| = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q_1 q_2}{d^2}$; $\varepsilon_0 = 8.854 \times 10^{-12} \underbrace{F m^{-1}}_{C^2 N^{-1} m^{-2}}$

is the vacuum permittivity.

SI units throughout this course

Electric field :



$$\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{q_{test}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{r}-\vec{r}'|^2} \hat{e}_{\vec{r}-\vec{r}'} = \frac{1}{4\pi\varepsilon_0} \frac{q(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

produced from 1 point charge q.

For a distribution of charges with charge density $\varrho(\vec{r}')$:

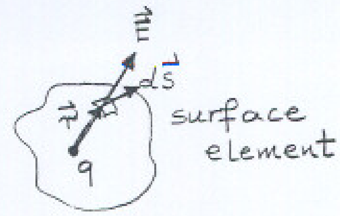
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3r' \frac{\varrho(\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad ; \quad [\varrho] = C\,m^{-3}$$

Since $\vec{\nabla}_r \dfrac{1}{|\vec{r}-\vec{r}'|} = -\dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$ (compare with $\dfrac{d}{dx}\dfrac{1}{x} = -\dfrac{1}{x^2}$),

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\varepsilon_0} \vec{\nabla}_r \int_V d^3r' \frac{\varrho(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\vec{\nabla}\phi(\vec{r}),$$

where $\phi(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_V d^3r' \frac{\varrho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ is the (static) scalar potential

Gauss' Law:



$$\oint_S \vec{E}\, d\vec{S} = \oint_S E\cos\theta\, dS = \oint_S \frac{1}{4\pi\varepsilon_0}\frac{q\cos\theta}{r^2}\, dS$$

$$= \frac{q}{4\pi\varepsilon_0}\oint_S d\Omega = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho(\vec{r})\, d^3r .$$

$\theta = \sphericalangle(\vec{E}, d\vec{S})$

$\vec{E} \parallel d^2\vec{r} = \hat{e}_{\vec{r}}\, r^2 d\Omega$

$\hat{e}_{\vec{r}}\, d\vec{S} = r^2 d\Omega$

$\cos\theta\, dS = r^2 d\Omega$

$\int d\Omega = 4\pi$

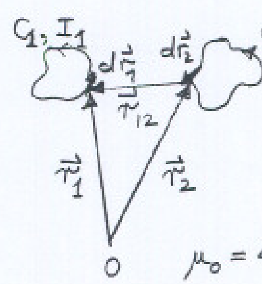
Also,
$$\oint_S \vec{E}\, d\vec{S} = \int_V \vec{\nabla}\vec{E}\, d^3r$$

Differential form of Gauss law:

$$\boxed{\vec{\nabla}\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (M1)}$$

1st Maxwell equation, as it also holds for time-varying electric and magnetic fields.

Magnetic force:



$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\vec{r}_1\times(d\vec{r}_2\times\vec{r}_{12})}{r_{12}^3}$$

is the force of current loop 2 on 1 (show that $\vec{F}_{21} = -\vec{F}_{12}$); $[I_{1,2}] = A$ (Ampère) is the current unit.

$\mu_0 = 4\pi\, 10^{-7}\,\frac{N}{A^2}$ is the vacuum permeability

Basic relation of constants:
$\varepsilon_0\mu_0 c^2 = 1$

$$\vec{F}_{12} = I_1 \oint d\vec{r}_1 \times \vec{B}(\vec{r}_1),$$

where $\vec{B}(\vec{r}_1)$ is the magnetic field induced by C_2:

$$\vec{B}(\vec{r}_1) = \frac{\mu_0 I_2}{4\pi}\oint_C \frac{d\vec{r}_2\times\vec{r}_{12}}{r_{12}^3} \quad : \text{Biot-Savart Law}$$

No free magnetic monopoles:

$$\boxed{\vec{\nabla}\vec{B} = 0 \qquad (M2)}$$ 2nd Maxwell equation

If one magnetic charge g existed, electric charge e would be quantized according to Dirac quantization condition

$$eg = \tfrac{1}{2}n \qquad (n = 1, 2, 3, \ldots)$$

(However, the Dirac realization faces difficulties with gauge invariance in Quantum Electrodynamics, see, e.g. Cheng+Li)

(M2) implies $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$, where $\vec{A}$ is the vector potential.

Indeed, $\vec{\nabla}\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$, i.e. div-curl vanishes.

$\vec{A}$ is not uniquely determined, i.e. for the change

$$\vec{A}(\vec{r},t) \longrightarrow \vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\chi(\vec{r},t),$$

$\vec{B}(\vec{r},t)$ remains invariant

The above change on $\vec{A}(\vec{r},t)$ is called gauge transformation and $\vec{B}(\vec{r},t)$ is gauge invariant.

$\longleftrightarrow$

Exercises: 1. Starting from the Biot-Savart law, show that

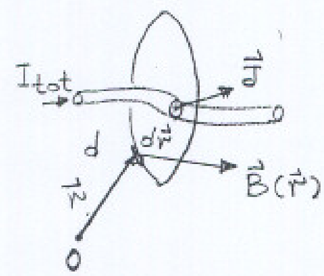
$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|} = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|},$$

where $\int_S \vec{J}\cdot d\vec{S} = I$, with $[\vec{J}] = A\,m^{-2}$, units of the current density.

2. Show that the vector potential for an homogenous constant magnetic field $\vec{B}$ is given by $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$. Is $\vec{A}$ unique?

Ampère's Law:

$$\oint_C \vec{B}\, d\vec{r} = \mu_0 I_{tot} = \mu_0 \int \vec{J}\, d\vec{S}$$



I_{tot}: total current through the closed loop, not current loop

$\vec{J}$: current density

Use of Stoke's theorem

$$\oint_C \vec{B}\, d\vec{r} = \int_S \vec{\nabla}\times\vec{B}\; d\vec{S}$$

$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$: Ampère's law in curl form.

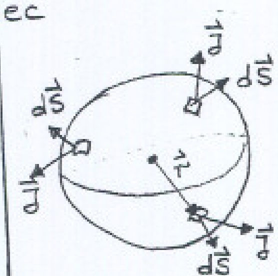
Ampère's law holds only for static $\vec{B}$ fields.

Charge conservation:

Start with the div. theorem applied to $\vec{J}$

$$\int_V \vec{\nabla}\vec{J}\, d^3r = \oint_S \vec{J}\, d\vec{S} = \text{loss of charge / sec from a closed surface}$$

$$= -\frac{\partial}{\partial t} q_{tot} = -\int \frac{\partial}{\partial t}\rho\; d^3r$$



Charge conservation requires:

$$\boxed{\vec{\nabla}\vec{J}(\vec{r},t) + \frac{\partial}{\partial t}\rho(\vec{r},t) = 0}$$

Since $\vec{\nabla}(\vec{\nabla}\times\vec{B}) = 0$, Ampère's law must read:

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\vec{E} \qquad \text{(M4)}}$$

4th Maxwell equation

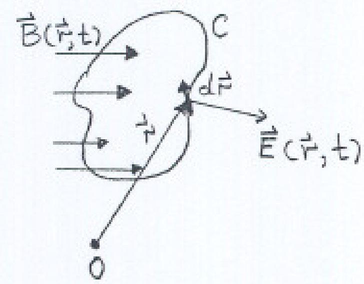
Exercise: Show that (M4) satisfies charge conservation.

Faraday's law:

$$\text{Electromotive Force (EMF)} = \oint_{C(S)} \vec{E}\, d\vec{r} = -\frac{\partial}{\partial t} \underbrace{\int_S \vec{B}\, d\vec{S}}_{\text{magnetic flux}}$$

Stoke's (curl) theorem:

$$\oint_C \vec{E}\, d\vec{r} = \int_S \vec{\nabla} \times \vec{E}\, d\vec{S} = -\int_S \frac{\partial}{\partial t}\vec{B}\, d\vec{S}$$



Time-varying $\vec{B}$-field through the loop C induces time-varying $\vec{E}$-field.

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \qquad \text{(M3)}}$$

3rd Maxwell equation

Remember by heart!

CURL GRAD VANISHES : $\vec{\nabla} \times \vec{\nabla} \Phi = 0$

DIV CURL VANISHES : $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

STOKE'S (CURL) THEOREM : $\oint_C \vec{E} \cdot d\vec{r} = \int_S \vec{\nabla} \times \vec{E}\, d\vec{S}$

GAUSS'S (DIV) THEOREM : $\int_V \vec{\nabla} \vec{E}\, d^3r = \oint_S \vec{E}\, d\vec{S}$

Exercises: 1. With the help of (M3), show that $\vec{E}$ is given by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

2. Show that $\vec{E}$ is (gauge) invariant under the gauge transformations of the potentials:

$$\Phi(\vec{r},t) \mapsto \Phi'(\vec{r},t) = \Phi(\vec{r},t) - \frac{\partial}{\partial t}\chi(\vec{r},t)$$
$$\vec{A}(\vec{r},t) \mapsto \vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\chi(\vec{r},t)$$

Summary of Maxwell's equations in vacuum:

(M1) $\vec{\nabla}\vec{E} = \frac{\varrho}{\varepsilon_0}$: Coulomb's and Gauss' law

(M2) $\vec{\nabla}\vec{B} = 0$: No free magnetic monopoles

(M3) $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$: Faraday's Law

(M4) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$: Ampère's law and charge conservation

- Charge conservation: $\vec{\nabla}\vec{J} + \frac{\partial}{\partial t}\varrho = 0$

- $\vec{E}$ and $\vec{B}$ fields: $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$
- Lorentz force on a moving charge with $\vec{v}$: $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$
- Gauge tranformations:

$$\left.\begin{aligned} \Phi &\mapsto \Phi' = \Phi - \frac{\partial}{\partial t}\chi \\ \vec{A} &\mapsto \vec{A}' = \vec{A} + \vec{\nabla}\chi \end{aligned}\right\} : \begin{array}{l}\vec{E} \text{ and } \vec{B} \\ \text{remain} \\ \text{unchanged}\end{array}$$

Coulomb gauge: $\vec{\nabla}\vec{A} = 0$
Lorentz gauge: $\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$
$\Big\}$: other choices are also possible, e.g. $\Phi = 0$ (radiation gauge)

$\longleftrightarrow$

Exercises: 1. Show that in a region free of charge and current, $\vec{E}$ and $\vec{B}$ fields satisfy the wave equations

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0 \, , \quad \nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = 0$$

2. Derive the potential field equations in the Lorentz gauge:

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = \frac{\varrho}{\varepsilon_0} \, , \quad \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \mu_0\vec{J} \, .$$

How those equations look like in other gauges?

Maxwell's Equations in materials:

Dielectrics (materials in $\vec{E}$ field)

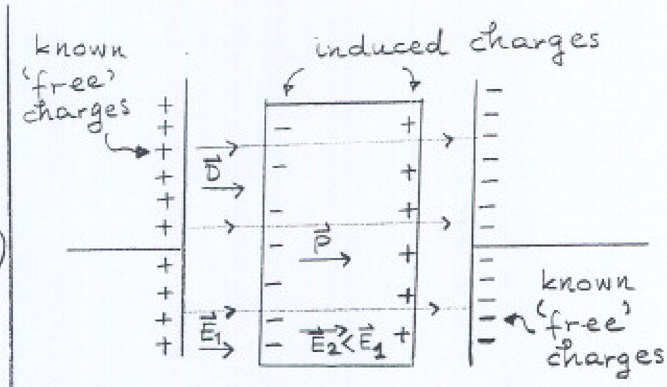
$$\vec{\nabla}\vec{E} = \frac{\varrho}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\varrho_{free} + \varrho_{ind}\right) \quad (M1)$$

If $\varrho_{ind} = -\vec{\nabla}\vec{P}$ ($\vec{P}$ is called polarization)

$$\vec{\nabla}\left(\varepsilon_0\vec{E} + \vec{P}\right) = \varrho_{free}$$

or $\boxed{\vec{\nabla}\vec{D} = \varrho_{free} \quad (\overline{M1})}$

with $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$, and



$\vec{D}$: Electric displacement. (field lines start and end on free charges)

$\vec{D} = \varepsilon_0\vec{E}_1 = \varepsilon_0\vec{E}_2 + \vec{P}$, where $\vec{P}$ is induced by the free charges.

$\vec{P} = \varepsilon_0\chi_E\vec{E}$; χ_E: electrical susceptibility (may be a tensor)

Then, $\vec{D} = (1+\chi_E)\varepsilon_0\vec{E} = \varepsilon_r\varepsilon_0\vec{E}$, where $\varepsilon_r = 1+\chi_E$ is the relative permittivity for linear media.

$(M2) = (\overline{M2})$; $(M3) = (\overline{M3})$ do not change in materials

Diamagnetics (materials in $\vec{B}$ field, e.g. in a solenoid)

As in the dielectrics, we have for $\frac{\partial\vec{E}}{\partial t} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} = \mu_0\left(\vec{J}_{free} + \vec{J}_{ind}\right) \quad (M4)$$

With $\vec{J}_{ind} = \vec{\nabla}\times\vec{M}$ ($\vec{M}$: magnetization),

$$\vec{\nabla}\times\left(\vec{B} - \mu_0\vec{M}\right) = \mu_0\vec{J}_{free}$$

or $\vec{\nabla}\times\vec{H} = \vec{J}_{free} \quad (\overline{M4})' \quad \left(\frac{\partial\vec{E}}{\partial t} = \frac{\partial\vec{D}}{\partial t} = 0\right)$

where $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ is the magetic intensity.

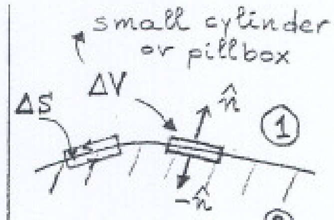
$\vec{M} = \chi_H\vec{H}$; χ_H: magnetic susceptibility (may be a tensor)

$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(1+\chi_H)\vec{H} = \mu_0\mu_r\vec{H}$; μ_r: relative permeability.

For $\frac{\partial\vec{E}}{\partial t} \neq 0$, $\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t} \quad (\overline{M4})}$

Behaviour of fields across material boundaries:

Boundary conditions for $\vec{E}$ and $\vec{D}$: (Dielectrics)



Electric and magnetic fields on boundaries between (1) and (2).

$(\overline{M1})$: $\vec{\nabla}\vec{D} = \rho_{free} \rightsquigarrow \int_{\Delta V} d^3r\, \vec{\nabla}\vec{D} = \int_{\Delta V} d^3r\, \rho_{free}$

$$\xrightarrow{\text{div. theorem}} \oint_{\Delta S} dS\, \hat{n}\vec{D} = \int_{\Delta V} dz\, dS\, \rho_{free} = \delta S\, \rho_s \, ,$$

where ρ_s is the surface density of free charge, $[\rho_s] = C\,m^{-2}$

Since $\oint_{\Delta S} dS\, \hat{n}\vec{D} = \delta S\, \hat{n}(\vec{D}_1 - \vec{D}_2) = \delta S\, \rho_s$,

we have $\boxed{\hat{n}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s}$. If $\rho_s = 0$, $\vec{D}_\perp$ is continuous.

For the $\vec{E}$ field, we have (for a surface loop ΔS)

$(M2)$: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \rightsquigarrow \int_{\Delta S} d\vec{S}\, \vec{\nabla}\times\vec{E} = -\int_{\Delta S} \frac{\partial\vec{B}}{\partial t} d\vec{S} \xrightarrow{\Delta S \to 0} 0$

$$\xrightarrow{\text{curl. theorem}} \oint_{\delta C} d\vec{r}\, \vec{E} = 0 \rightsquigarrow \boxed{\vec{E}_{1\parallel} - \vec{E}_{2\parallel} = 0}$$

. $\vec{E}_\parallel$ is always continuous

$\longleftrightarrow$

Exercises: 1. For a diamagnetic boundary, show that

$\boxed{\vec{B}_{1\perp} - \vec{B}_{2\perp} = 0}$, i.e. $\vec{B}_\perp$ is continuous.

(Hint: Consider (M2) $\vec{\nabla}\vec{B} = 0$ in a pillbox)

2. Prove the diamagnetic boundary condition for $\vec{B}$:

$\boxed{\hat{n}\times(\vec{H}_1 - \vec{H}_2) = \vec{J}_s}$; $\vec{J}_s$: surface current per unit width. $[\vec{J}_s] = A\,m^{-1}$

Show that if $\vec{J}_s = 0$, $\vec{H}_\parallel$ is continuous.

(Hint: Consider $(\overline{M4})$ on a surface loop)

Exercise on the delta function:

In the following exercise we show that Dirac's delta function is given by the formula below:

$$\delta(x) = \frac{1}{\sqrt{\pi}} \lim_{a \to 0} \frac{1}{a} \exp\left(-x^2/a^2\right), \qquad a > 0 \tag{1}$$

Prove the following properties of the delta function:

$$(i) \qquad \int_{-\infty}^{\infty} dx\; \delta(x) = 1 \tag{3}$$

$$(ii) \qquad \int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0) \tag{2}$$

$$(iii) \qquad \delta(cx) = \frac{1}{|c|}\, \delta(x), \qquad c \neq 0 \tag{4}$$

$$(iv) \qquad x\,\delta(x) = 0 \tag{5}$$

$$(v) \qquad \int_{-\infty}^{\infty} dx\; \delta'(x)\, f(x) = -f'(0) \tag{6}$$

$$(vi) \qquad \delta\left(x^2 - c^2\right) = \frac{1}{2\,|c|}\left(\delta(x-c) + \delta(x+c)\right) \tag{7}$$

$$(vii) \qquad \delta(f(x)) = \sum_i \frac{1}{\left|df/dx\right|_{x=x_i}}\, \delta(x - x_i). \tag{8}$$

Here x_i are the roots of the function $f(x)$ $(f(x_i) = 0)$ within the integration intervals.

$$(viii) \qquad \int_{-\infty}^{x} dy\,\delta(y-c) = \Theta(x-c) = \begin{cases} 0 & for\;\; x < c \\ 1 & for\;\; x > c \end{cases} \tag{9}$$

$$(ix) \qquad \frac{d\,\Theta(x-c)}{dx} = \delta(x-c) \tag{10}$$

Hint: Use the following Gauss integral:

$$\int_{-\infty}^{\infty} dx\; \exp\left(-x^2/a^2\right) = \sqrt{\pi} a\,, \tag{11}$$

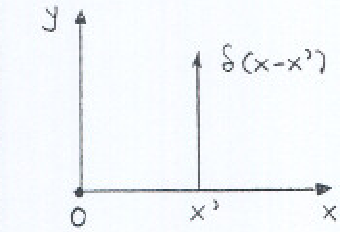
and the Taylor expansion

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \cdots\cdots \tag{12}$$

Point charge and Dirac's δ-function:

Electron may be a point charge, without structure (exp. $< 10^{-20}\,m$)

Introduce the Dirac δ-function to describe charge density



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Height : + ∞

$$\delta(x-x') = \begin{cases} +\infty, & x = x' \\ 0, & x \neq x' \end{cases}$$

and $\displaystyle\int_{-\infty}^{+\infty} \delta(x-x')\,dx = 1$ or $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\delta(x-x')\,dx = f(x')$

In 3-D: $\delta^{(3)}(\vec{r}-\vec{r}\,') \triangleq \delta(x-x')\,\delta(y-y')\,\delta(z-z')$,

$$\Longrightarrow \int_{V\to\infty} d^3r\, f(\vec{r})\,\delta^{(3)}(\vec{r}-\vec{r}\,') = f(\vec{r}\,')$$

Therefore, point charge density: $\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}-\vec{r}_0)$, situated at $\vec{r}_0$

Consider the (static) scalar potential $\Phi = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r}$; $(\vec{r}_0 = 0)$

Then, $\vec{E} = -\vec{\nabla}\Phi$ and $\vec{\nabla}\vec{E} = \dfrac{\rho(\vec{r})}{\varepsilon_0}$ (M1),

or equivalently $\nabla^2\Phi = -\dfrac{\rho}{\varepsilon_0} \Rightarrow -\dfrac{1}{4\pi}\nabla^2\dfrac{1}{r} = \delta^{(3)}(\vec{r})$

Convince yourself that in spherical coordinates:

$$\nabla^2\frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right) = 0, \text{ for } r>0$$

In general, we have

$$\boxed{-\frac{1}{4\pi}\nabla_r^2\frac{1}{|\vec{r}-\vec{r}\,'|} = \delta^{(3)}(\vec{r}-\vec{r}\,')}$$

Remark: $\nabla_r^2 G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$; $G(\vec{r},\vec{r}\,') = -\dfrac{1}{4\pi}\dfrac{1}{|\vec{r}-\vec{r}\,'|} + f(\vec{r},\vec{r}\,')$, with $\nabla_r^2 f(\vec{r},\vec{r}\,') = 0$, is the Green function without boundary condition.

I.2 Laplace and Poisson Equations

The Poisson equation: $\boxed{\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}}$ in electrostatics

→ $\Phi(\vec{r})$ may be found with the help of Green's functions; see p.(I.1.1) for a general solution without special boundary conditions:

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \qquad \text{(Which is the Green function?)}$$

$\Phi(\vec{r}) \xrightarrow{\vec{r}\to\infty} 0$, if $\rho(\vec{r})$ is spatially confined and no other charges are present.

This, however, is not always the case!

(see below discussion of other methods)

The Laplace equation: $(\rho = 0)$ $\boxed{\nabla^2 \Phi = 0}$

- $\Phi(\vec{r})$ is unique (except for an additive constant), if it satisfies the Dirichlet and/or Neumann boundary conditions on on an enclosing surface $S(V)$, namely
  - (a) $\Phi(\vec{r})$ is known on $S(V)$: Dirichlet
  - (b) Normal derivative $\frac{\partial\Phi}{\partial n} \equiv \hat{n}\cdot\vec{\nabla}\Phi$ on $S(V)$ is given : Neumann
  - (c) Mixed condition, with $\alpha\Phi(\vec{r}) + \beta\frac{\partial\Phi}{\partial n}(\vec{r})$ known on $S(V)$

- $\Phi(\vec{r})$ may easily be determined by exploiting the symmetry of the problem.

  Need to discuss the solutions of the Laplace equation in rectangular (or cartesian), cylindrical polar and spherical polar coordinates.