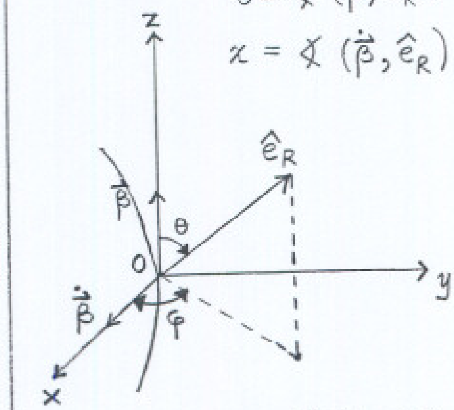
Radiation by a charged particle with $\vec{\beta} \perp \dot{\vec{\beta}}$:

$\vec{\alpha} = c\dot{\vec{\beta}}$

$\theta = \measuredangle(\vec{\beta}, \hat{e}_R)$

$\chi = \measuredangle(\dot{\vec{\beta}}, \hat{e}_R)$



$\cos\chi = \sin\theta\cos\varphi$

Start from the general formula for $\frac{dP}{d\Omega}$:

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\,\frac{\left[\hat{e}_R \times ((\hat{e}_R - \vec{\beta}) \times \dot{\vec{\beta}}\,\right]^2}{(1 - \vec{\beta}\,\hat{e}_R)^5},$$

where all quantities are evaluated at the retarded time.

Calculate the numerator first:

$$\begin{aligned}
\hat{e}_R \times \left((\hat{e}_R - \vec{\beta}) \times \dot{\vec{\beta}}\right) &= \hat{e}_R \times (\hat{e}_R \times \dot{\vec{\beta}}) - \hat{e}_R \times (\vec{\beta} \times \dot{\vec{\beta}}) \\
&= (\hat{e}_R \cdot \dot{\vec{\beta}})\,\hat{e}_R - \dot{\vec{\beta}} - (\hat{e}_R \cdot \dot{\vec{\beta}})\,\vec{\beta} + (\hat{e}_R \cdot \vec{\beta})\,\dot{\vec{\beta}} \\
&= (\hat{e}_R \cdot \dot{\vec{\beta}})(\hat{e}_R - \vec{\beta}) - \dot{\vec{\beta}}\left(1 - (\hat{e}_R \cdot \vec{\beta})\right),
\end{aligned}$$

and

$$\begin{aligned}
\left[\hat{e}_R \times \left((\hat{e}_R - \vec{\beta}) \times \dot{\vec{\beta}}\right)\right]^2 &= (\hat{e}_R \cdot \dot{\vec{\beta}})^2 (\hat{e}_R - \vec{\beta})^2 - 2(\hat{e}_R \dot{\vec{\beta}})^2 \left(1 - (\hat{e}_R \cdot \vec{\beta})\right) \\
&\qquad + \dot{\beta}^2 \left(1 - (\hat{e}_R \cdot \vec{\beta})\right)^2 \\
&= -(1 - \beta^2)(\hat{e}_R \cdot \dot{\vec{\beta}})^2 + \dot{\beta}^2 \left(1 - (\hat{e}_R \cdot \vec{\beta})\right)^2 .
\end{aligned}$$

The radiated power $dP_\perp/d\Omega$ for $\vec{\beta} \perp \dot{\vec{\beta}}$ (synchrotron radiation) is

$$\boxed{\frac{dP_\perp}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\,\frac{\dot{\beta}^2}{(1 - \beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2(1 - \beta\cos\theta)^2}\right]} \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

$\longleftrightarrow$

Exercises:

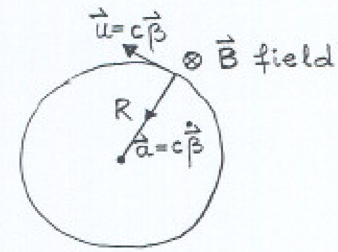
1. Show that there is no synchrotron radiation for angles $\varphi = 0$ and $\theta_{min} = \cos^{-1}\beta$, while the maximum of radiation occurs for $\varphi = 0$ and $\theta_{max} = 0$.

2. Integrate $dP_{\parallel}/d\Omega$ and $dP_\perp/d\Omega$ over Ω to obtain

$$\boxed{P_{\parallel} = \gamma^6 P_L\ ,\ P_\perp = \gamma^4 P_L\ ;\ P_L = \frac{q^2\dot{\beta}^2}{6\pi\varepsilon_0 c}}$$

Charged particle in a circular orbit:

CERN: LHC (2008) $E_{tot} = 14\,\text{TeV}$



Particle acceleration in a circular orbit of radius R.

High-energy colliders, such as the LEP2 and LHC at CERN and Tevatron at FERMILAB, accelerate particles in a circular orbit.

The accelerated particles are kept in a circular orbit by applying an external $\vec{B}$ field as shown in the figure.

Employing the fact that $a = \frac{u^2}{R}$, the total radiated power $P_\perp$ is

$$P_\perp = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}\gamma^4 = \frac{q^2 c \beta^4}{6\pi\varepsilon_0 R^2}\gamma^4 .$$

Equating the centripetal force $\vec{F}_c$ with the Lorentz force $\vec{F}_L$ yields

$$\vec{F}_c \mathrel{\hat{=}} \gamma m \vec{a} = q\vec{u}\times\vec{B} \mathrel{\hat{=}} \vec{F}_L \rightsquigarrow \gamma m \frac{u^2}{R} = q u B \rightsquigarrow \boxed{R = \frac{m\gamma\beta c}{qB}}$$

Then, the total radiated power $P_\perp$ becomes

$$\boxed{P_\perp = \frac{q^4 B^2 \beta^2 \gamma^2}{6\pi\varepsilon_0 m^2 c}}$$

m: particle's mass

B: applied magnetic field

Remarks:

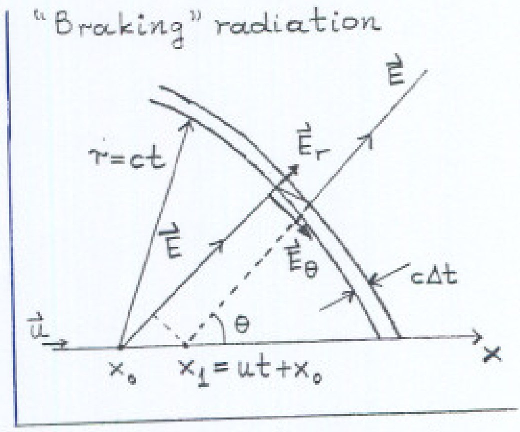
- Circular accelerators are rather efficient in the sense that the particles pass repeatedly through the same accelerating sections.
- The maximum guide $\vec{B}$-field that minimizes R is fixed by the available magnet technology, e.g. superconducting magnets at LHC.
- The synchrotron radiation is more severe for lighter particles, e.g. electrons than protons. (How much ?).

$\longleftrightarrow$

Exercise: Show that the radiation from a circular moving particle is by a factor γ^2 larger than the radiation from a linearly accelerated particle for the same applied force $\vec{F} = \frac{d}{dt}\vec{p}$, with $\vec{p} = m\gamma\vec{u}$, i.e. show that

$$P_{\parallel} = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3}\left|\frac{d\vec{p}}{dt}\right|^2 \quad \text{and} \quad P_\perp = \frac{q^2\gamma^2}{6\pi\varepsilon_0 m^2 c^3}\left|\frac{d\vec{p}}{dt}\right|^2 .$$

A geometric description of non-relativistic Bremsstrahlung:



Consider a charge point $+q$ with velocity $\vec{u}$ being brought rapidly to rest in a foil radiator at $t=0$, $x=x_0$.

Stopping time Δt and deceleration $\vec{a}=-\frac{\vec{u}}{\Delta t}$.

At time t:

(i) for $r>ct$, $-\vec{E}$ points to $x_1=x_0+ut$ (the anticipated present position)

(ii) for $r<ct$, $-\vec{E}$ points to x_0, i.e. $\vec{E}=\frac{q}{4\pi\varepsilon_0 r^2}\hat{e}_r=\vec{E}_r$

The disturbance travels at speed of light in a ripple with width $c\Delta t$. Also, lines of $\vec{E}$ are continuous, as there are no charges in free space.

Geometric estimate of radiation fields:

$$\vec{E}_r \sim c\Delta t\ ,\quad \vec{E}_\theta \sim (x_1-x_0)\sin\theta \quad \text{and} \quad \frac{E_\theta}{E_r}=\frac{(x_1-x_0)\sin\theta}{c\Delta t}=\frac{ut\sin\theta}{c(u/a)}$$

$$\text{Then,}\quad E_\theta=\frac{at\sin\theta}{c}E_r=\frac{q}{4\pi\varepsilon_0 r^2}\,\frac{a\overbrace{(r/c)}^{=t}\sin\theta}{c}=\underline{\frac{q}{4\pi\varepsilon_0 c^2}\,\frac{a\sin\theta}{r}}$$

$E_\theta \propto \frac{1}{r}$ is the accelerating (radiation) field.

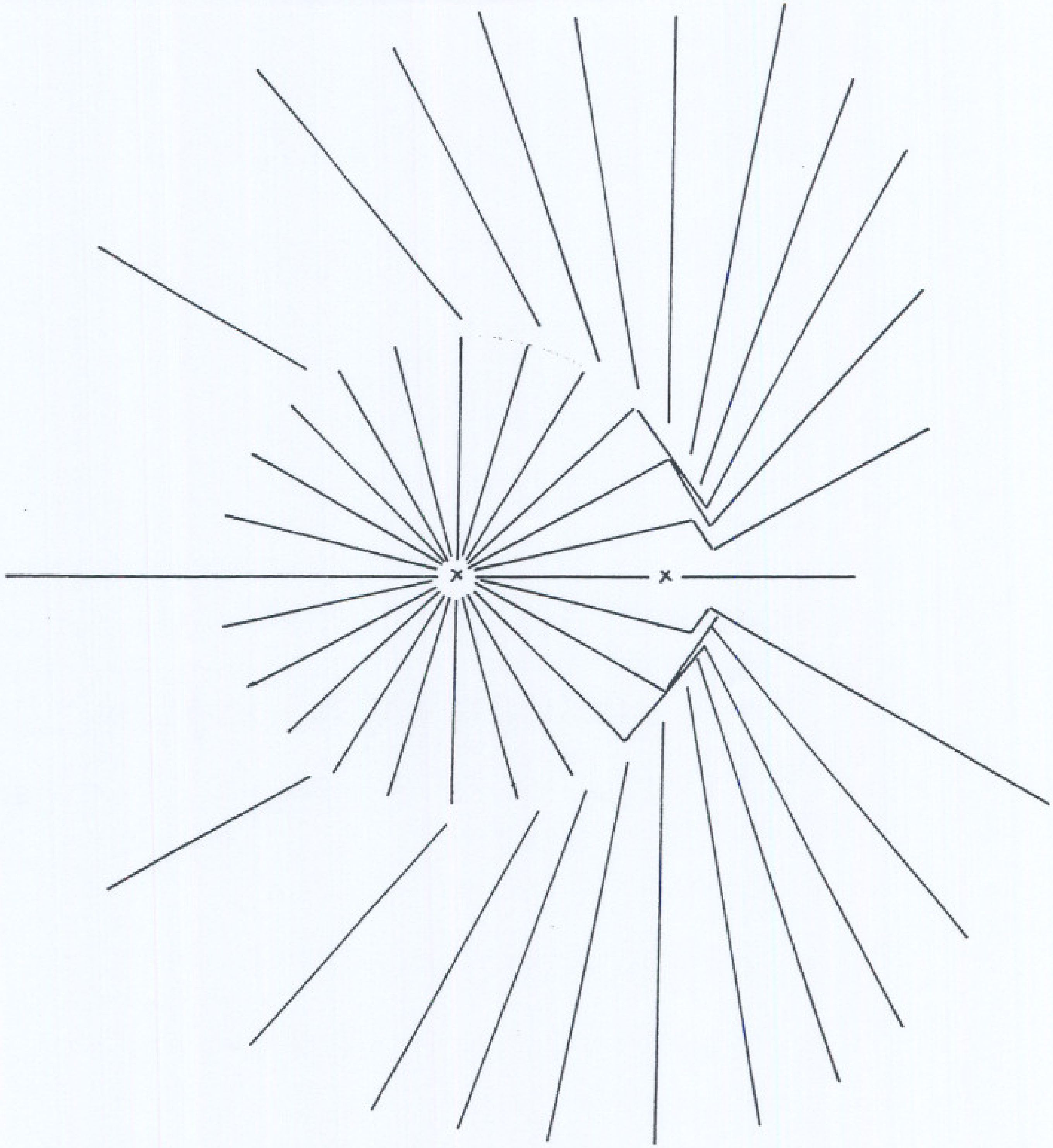
Poynting vector: $|\vec{S}|=\frac{1}{\mu_0}E_\theta B_\theta=\frac{1}{\mu_0 c}E_\theta^2$

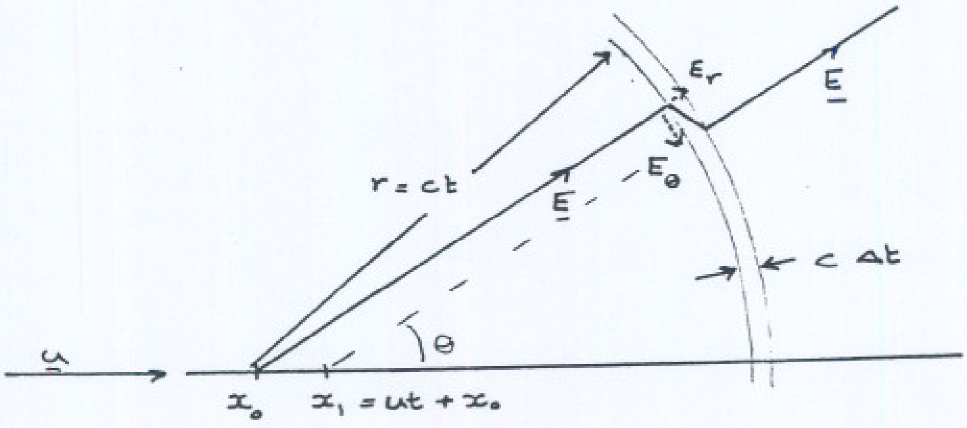
and the power radiated per unit solid angle is

$$\frac{dP}{d\Omega}=\frac{1}{\mu_0 c}E_\theta^2 r^2=\frac{q^2}{16\pi^2\varepsilon_0 c^3}a^2\sin^2\theta\,.$$

The above formula is identical to the differential Larmor's formula $dP_L/d\Omega$ on page (II.3.4).

RADIATION FROM ACCELERATED CHARGES : BREMSSTRAHLUNG





SUMMARY

PC3642 (i) Relativistic particle, $\mathbf{a}$ collinear with $\mathbf{u}$:

$$\text{Power radiated } \frac{dP_{\parallel}(t_R)}{d\Omega} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{a^2}{\mu_0 c^5} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \quad ; \quad \begin{array}{|l} \varepsilon_0\mu_0 c^2 = 1 \\ a = c\dot{\beta} \end{array}$$

The denominator directs power forwards in a cone; $\theta_{max} \approx 1/2\gamma$ (see Examples III, Qu.3). The radiation is polarised with the $\mathbf{E}$-vector lying in the plane containing $\mathbf{a}$ and the field point.

$$\text{Total power}_{\parallel} = \int_{4\pi} \frac{dP_{\parallel}}{d\Omega} d\Omega = \gamma^6 \times \overbrace{\text{Larmor formula}}^{: P_L = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3}}$$

An application: Photonuclear reaction studies (structure of nucleus, structure of nucleon). High energy electrons from a LINAC (850 MeV at Mainz accelerator) are decelerated by a foil *radiator* to produce a highly-focused beam of high-energy photons which are allowed to strike the experimental target. Each photon in the beam may be *tagged* with a measurement of the energy of the corresponding electron after it has scattered from the radiator.

(ii) Relativistic particles, $\mathbf{a}$ perpendicular to $\mathbf{u}$: Synchrotron radiation:

$$\text{Power radiated } \frac{dP_{\perp}(t_R)}{d\Omega} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{a^2}{\mu_0 c^5} \frac{\left[(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi\right]}{(1-\beta\cos\theta)^5}$$

The denominator distorts the doughnut pattern into a beam strongly focused in the direction of $\mathbf{u}$. Beam angular width $\sim 1/\gamma$.

$$\text{Total power}_{\perp} = \int_{4\pi} \frac{dP_{\perp}}{d\Omega} d\Omega = \gamma^4 \times \underbrace{\text{Larmor formula}}_{: P_L}$$

* The radiation is highly polarised with the $\mathbf{E}$-vector lying in the plane containing $\mathbf{a}$ and the field point.

* The frequency spectrum for this (synchrotron) radiation [and for the collinear case in (i)] can be calculated by Fourier techniques relating the radiation as a function of time t to a function of frequency ω.

* The synchrotron radiation limits the performance of charged particle accelerators (e.g. LEP at CERN). It is essential to reduce **a**; the accelerator must have a large radius of curvature. The 60 GeV LEP ring has a radius of 4.2 km, where the 14 TeV LHC will operate.

* Before the discovery of the Crab pulsar it was known that the visible light from the nebula was from synchrotron radiation because it was highly polarised. Analysis of the light reveasl properties of the magnetic field throughout the nebula.

* Synchrotron radiation has many applications in physics, chemistry, biology research. The Daresbury Laboratory 2 GeV electron synchrotron radiation source (SRS) (and in later years, DIAMOND at the Rutherford Lab) offers controllable radiation in the UV/x-ray region with very short pulse structure and highly-polarised. Although collinear acceleration produces more power (by a factor of γ^6 rather than γ^4) it is far easier to accelerate relativistic particles radially.

Daresbury SRS The electrons are initially accelerated by a LINAC, raised to 600 MeV by a booster and injected into the 2 GeV storage ring in a bunch. The ring is constructed like a polygon with straight sections and sharp corners. Experimental beam lines project tangentially from the ring at the corners. Wavelength-selection devices (gratings) may be used in the beam lines. Periodic magnet structures (*undulators, wigglers*) may be used in the straight sections of the storage ring to tailor radiation for specific purposes.

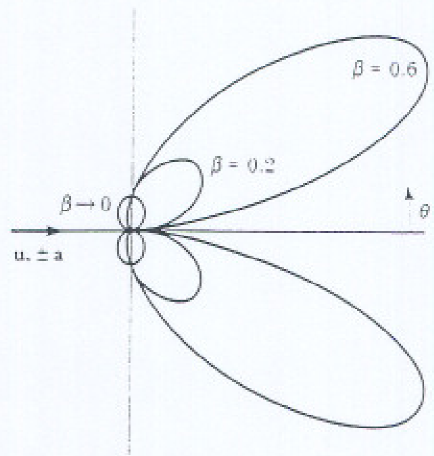


FIGURE 8-6. Radiation from accelerated charge with **u** parallel to **a**.

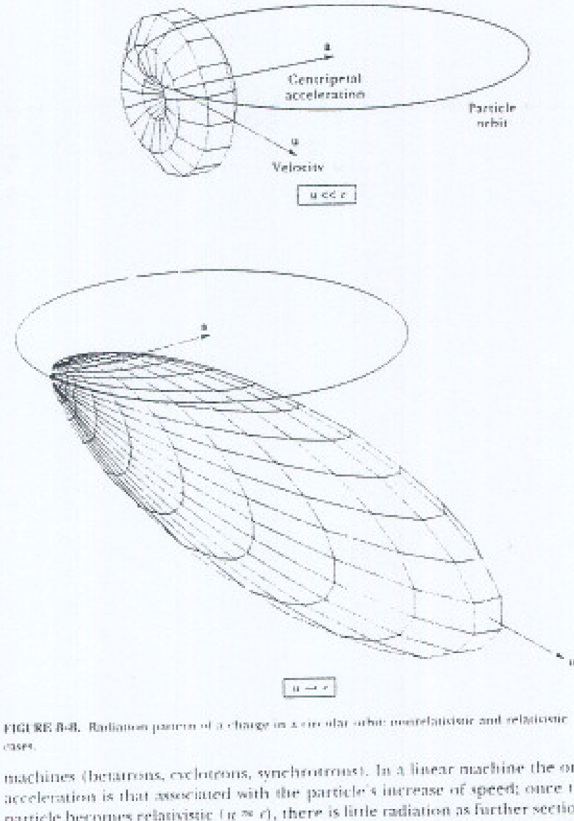


FIGURE 8-8. Radiation pattern of a charge in a circular orbit: nonrelativistic and relativistic cases.

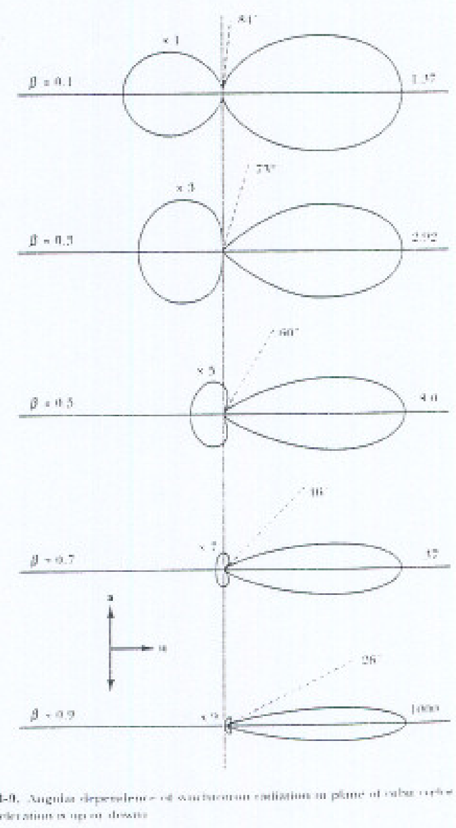
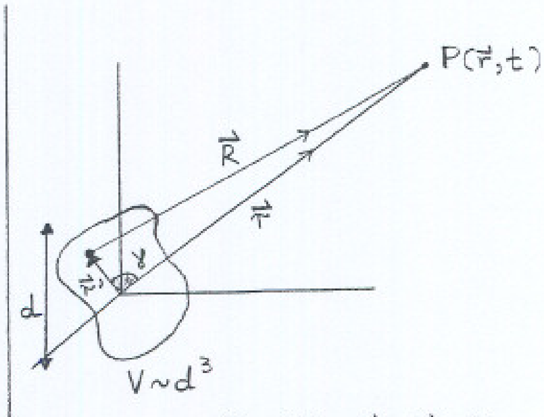


FIGURE 8-9. Angular dependence of synchrotron radiation in plane of orbit; velocity is to the right; acceleration is up or down.

machines (betatrons, cyclotrons, synchrotrons). In a linear machine the only acceleration is that associated with the particle's increase of speed; once the particle becomes relativistic ($u \approx c$), there is little radiation as further sections of the accelerator continue to increase the particle's energy. In circular machines, there is a continuous centripetal acceleration, and its associated radiation becomes dominant as $u \to c$. For a relativistic particle in a circular orbit of radius ρ, the centripetal acceleration approaches

III. Harmonically Varying Sources



$\gamma = \sphericalangle(\vec{r}\,',\vec{r})$, $\vec{R} = \vec{r} - \vec{r}\,'$

$t_{ret} = t - R/c$

Consider the harmonic charge and current densities:

$$\left.\begin{aligned} \rho(\vec{r}\,',t') &= \rho(\vec{r}\,')\, e^{-i\omega t'} \\ \vec{J}(\vec{r}\,',t') &= \vec{J}(\vec{r}\,')\, e^{-i\omega t'}, \end{aligned}\right\}: \text{only real parts are physical!}$$

which satisfy the continuity equation

$$\vec{\nabla}_{r'}\, \vec{J}(\vec{r}\,',t') + \frac{\partial}{\partial t'}\rho(\vec{r}\,',t') = 0 \,.$$

$\longleftrightarrow$

Assumptions in radiation zone approximation (RZA):

$$\boxed{r' \ll \lambda \ll R}\,, \quad \text{where} \quad \lambda = 2\pi\frac{c}{\omega} \qquad \left(k = \frac{2\pi}{\lambda} = \frac{\omega}{c}\right)$$

In RZA, $R = |\vec{r} - \vec{r}\,'| = (r^2 + r'^2 - 2rr'\cos\gamma)^{1/2} = r\left(1 - 2\frac{r'}{r}\cos\gamma\right)^{1/2} + \mathcal{O}(r'^2)$

$$= r - r'\cos\gamma + \mathcal{O}(r'^2) = r - \vec{r}\,'\cdot\hat{e}_r + \mathcal{O}(r'^2) \,.$$

RZA may well apply to atomic and nuclear transitions as well as to pulsars.

$\longleftrightarrow$

The time retarded scalar and vector potentials are

$$\Phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\, \frac{\rho(\vec{r}\,')\, e^{-i\omega(t-R/c)}}{R} = \frac{e^{-i\omega t}}{4\pi\varepsilon_0}\int_V d^3r'\, \frac{\rho(\vec{r}\,')\, e^{ikR}}{R} \mathrel{\hat{=}} e^{-i\omega t}\,\Phi(\vec{r}),$$

with $k = \frac{\omega}{c}$, and

$$\vec{A}(\vec{r},t) = \frac{\mu_0\, e^{-i\omega t}}{4\pi}\int_V d^3r'\, \frac{\vec{J}(\vec{r}\,')\, e^{ikR}}{R} \mathrel{\hat{=}} e^{-i\omega t}\,\vec{A}(\vec{r}) \,.$$

Note that all time dependence in Φ and $\vec{A}$ factors out.

$\longleftrightarrow$

Exercise: Show that the above $e^{-i\omega t}$- dependent charge and current densities are related through

$$\rho(\vec{r}\,',t') = \frac{-i}{\omega}\vec{\nabla}_{\vec{r}\,'}\vec{J}(\vec{r}\,',t')\,, \text{ and hence } \rho(\vec{r}\,') = \frac{-i}{\omega}\vec{\nabla}_{\vec{r}\,'}\vec{J}(\vec{r}\,')$$

Radiation fields in the RZA:

The $\vec{B}$-field is calculated by

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) = e^{-i\omega t}\,\vec{\nabla}\times\vec{A}(\vec{r}) \triangleq e^{-i\omega t}\,\vec{B}(\vec{r}),$$

with $\underline{\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})}$.

The $\vec{E}$-field can be determined from (M4):

$$\vec{\nabla}\times\vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t)$$ (since $\vec{J}(\vec{r})=0$ at the observation point P)

With $\vec{E}(\vec{r},t) = e^{-i\omega t}\vec{E}(\vec{r})$, we get

$$e^{-i\omega t}\,\vec{\nabla}\times\vec{B}(\vec{r}) = -i\frac{\omega}{c^2}e^{-i\omega t}\,\vec{E}(\vec{r}) \rightsquigarrow \underline{\vec{E}(\vec{r}) = i\frac{c}{k}\,\vec{\nabla}\times\vec{B}(\vec{r})}\,. \quad \left(k=\frac{\omega}{c}\right)$$

We see that $\underline{\vec{E}}$ and $\underline{\vec{B}}$ can be entirely determined by $\underline{\vec{A}(\vec{r})}$.

In the RZA, $\vec{A}(\vec{r})$ takes the simpler form

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_V d^3r'\, e^{-ik\vec{r}\,'\cdot\hat{e}_r}\,\vec{J}(\vec{r}\,') \triangleq \frac{\mu_0}{4\pi}\underbrace{\frac{e^{ikr}}{r}}_{\text{outgoing spherical wave term}}\vec{f}(\hat{e}_r)$$

$$\rightsquigarrow \boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{f}(\hat{e}_r)}\,, \text{ with } \underline{\vec{f}(\hat{e}_r) = \int_V d^3r'\, e^{-ik\vec{r}\,'\cdot\hat{e}_r}\,\vec{J}(\vec{r}\,')}$$

The radiation (accelerating) $\vec{B}$-field is calculated by

$$\vec{B}_a = \vec{\nabla}\times\vec{A}(\vec{r})\Big|_a = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\vec{f}(\hat{e}_r)\right)$$
$$= \frac{\mu_0}{4\pi}\left[\underbrace{\frac{e^{ikr}}{r}\vec{\nabla}\times\vec{f}(\hat{e}_r)}_{\propto\frac{1}{r^2}} + \underbrace{\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)}_{\hat{e}_r ik\frac{e^{ikr}}{r}+O\left(\frac{1}{r^2}\right)}\times\vec{f}(\hat{e}_r)\right]$$

Thus, we have $$\boxed{\vec{B}_a = ik\,\hat{e}_r\times\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\,ik\,\frac{e^{ikr}}{r}\left(\hat{e}_r\times\vec{f}(\hat{e}_r)\right)}$$

Exercise: Show that the corresponding radiation $\vec{E}$ field is given by

$$\boxed{\vec{E}_a = -c\,\hat{e}_r\times\vec{B}_a = \frac{\mu_0 c}{4\pi}\,ik\,\frac{e^{ikr}}{r}\left[\vec{f} - (\hat{e}_r\cdot\vec{f})\,\hat{e}_r\right]}$$

III.1 Electric Dipole Radiation

All information of the type of radiation is encoded in the source function

$$\vec{f}(\hat{e}_r) = \int_V d^3r' \, e^{-ik\vec{r}'\cdot\hat{e}_r} \, \vec{J}(\vec{r}')$$

In the RZA, $\underline{r' \ll \lambda} \ll R \;\rightsquigarrow\; \frac{r'}{\lambda} \ll 1 \;\rightsquigarrow\; \underline{kr' \ll 1}$.

So, we can expand $e^{-ik\vec{r}'\cdot\hat{e}_r} = 1 - ik\vec{r}'\cdot\hat{e}_r + \cdots$

1st term '1' will give electric dipole radiation: E1 radiation

2nd term '$-ik\vec{r}'\cdot\hat{e}_r$' will give magnetic dipole radiation : M1 radiation, and electric quadrupole radiation : E2 radiation.

Here, we consider the 1st term only:

$$\vec{f}^{(1)} = \int_V d^3r' \, \vec{J}(\vec{r}') \neq 0 \text{ , since } \vec{\nabla}_{\vec{r}'}\cdot\vec{J}(\vec{r}') = +i\omega\rho(\vec{r}') \neq 0$$

(see exercise on p. III.1.1)

x-component :
$$f_x^{(1)} = \int_V d^3r' \, \hat{e}_x\cdot\vec{J}(\vec{r}') = \int_V d^3r' \, \underbrace{\left(\vec{\nabla}_{r'} x'\right)}_{=\hat{e}_x}\cdot\vec{J}(\vec{r}')$$

$$= \int_V d^3r' \left[\vec{\nabla}_{r'}\cdot\left(x'\vec{J}(\vec{r}')\right) - x' \underbrace{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}')}_{+i\omega\rho(\vec{r}')} \right]$$

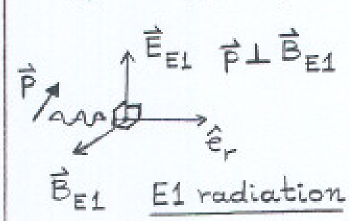
$$= \oint_{S(V)} \underbrace{d\vec{S}'\cdot\vec{J}(\vec{r}')}_{=0 \text{ on } S} x' + i\omega \underbrace{\int_V d^3r' \, x'\rho(\vec{r}')}_{\substack{\text{x-component of} \\ \text{the electric dipole } \vec{p}}} = -i\omega \, p_x$$

Consequently, $\vec{f}^{(1)} = -i\omega\vec{p}$.

From p. (III.1.2) (including $e^{-i\omega t}$ dependence), we get

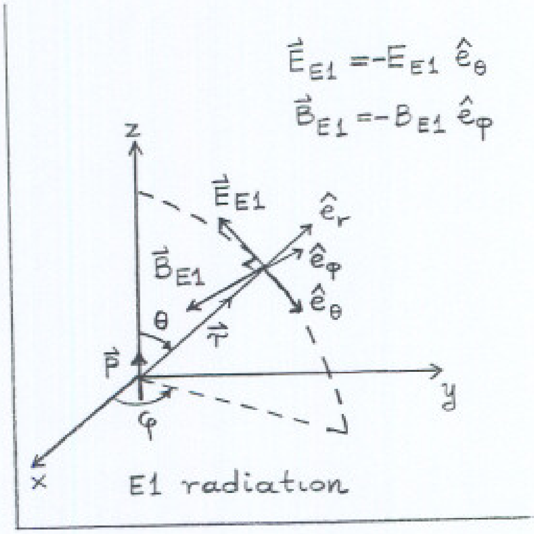
$$\boxed{\begin{aligned} \vec{B}_{E1}(\vec{r},t) &= \frac{\mu_0}{4\pi c}\,\omega^2\,\frac{e^{i(kr-\omega t)}}{r}\,(\hat{e}_r\times\vec{p}) \\ \vec{E}_{E1}(\vec{r},t) &= \frac{\mu_0}{4\pi}\,\omega^2\,\frac{e^{i(kr-\omega t)}}{r}\,\Big[\underbrace{\vec{p} - (\hat{e}_r\cdot\vec{p})\hat{e}_r}_{=-\hat{e}_r\times(\hat{e}_r\times\vec{p})}\Big] \end{aligned}}$$

$\vec{E}_{E1} \perp \vec{B}_{E1} \perp \hat{e}_r$



E1 radiation

E1 radiation:



We take $\vec{p} = p\,\hat{e}_z$, and in spherical polars

$$\hat{e}_r \times \vec{p} = -p\sin\theta\,\hat{e}_\varphi$$

$$\vec{p} - (\hat{e}_r\cdot\vec{p})\,\hat{e}_r = (\hat{e}_r\times\vec{p})\times\hat{e}_r = -p\sin\theta\;\hat{e}_\varphi\times\hat{e}_r$$
$$= -p\sin\theta\,\hat{e}_\theta\,.$$

Thus, $$\vec{B}_{E1} = -\frac{\mu_0}{4\pi c}\,\frac{e^{ikr}}{r}\,\omega^2 p\, e^{-i\omega t}\sin\theta\,\hat{e}_\varphi$$

and $$\vec{E}_{E1} = -\frac{\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\,\omega^2 p\, e^{-i\omega t}\sin\theta\,\hat{e}_\theta\,.$$

The Poynting vector due to the E1 radiation fields is

$$\vec{S} = \frac{1}{\mu_0}\,\mathrm{Re}(\vec{E}_{E1})\times\mathrm{Re}(\vec{B}_{E1}) = \frac{c}{\mu_0}\left[\mathrm{Re}(B_{E1})\right]^2\hat{e}_r$$

and
$$|\vec{S}| = \hat{e}_r\cdot\vec{S}(t) = \frac{\mu_0}{16\pi^2 c}\,\frac{\omega^4 p^2}{r^2}\sin^2\theta\,\cos^2(kr-\omega t)\,.$$

The time-average Poynting vector is

$$\langle|\vec{S}|\rangle \mathrel{\hat{=}} \frac{1}{T}\int_0^T |\vec{S}|\,dt = \frac{\mu_0}{16\pi^2 c}\,\frac{\omega^4 p^2}{r^2}\sin^2\theta\,\underbrace{\langle\cos^2(kr-\omega t)\rangle}_{=\frac{1}{2}\text{ (see exercise 1 below)}}\,,$$

where T is the period of oscillation: $T = \frac{2\pi}{\omega}$.

The time-average power radiated by an oscillating electric dipole per unit solid angle is

$$\boxed{\left\langle\frac{dP_{E1}}{d\Omega}\right\rangle = r^2\langle|\vec{S}|\rangle = \frac{1}{32\pi^2\varepsilon_0 c^3}\,\omega^4 p^2\sin^2\theta}$$

Exercises: 1. Show that $\langle\cos^2(kr-\omega t)\rangle = \frac{1}{T}\int_0^T\cos^2(kr-\omega t)\,dt = \frac{1}{2}$

2. Show that the time-average total power is

$$\boxed{\langle P_{E1}\rangle = \frac{\omega^4 p^2}{12\pi\,\varepsilon_0 c^3}}\,.$$

How does the formula for $\langle P\rangle$ compare with Larmor's formula on p.(II.3.5)?

III.2 Magnetic Dipole Radiation

We now turn our attention to the 2^{nd} term of $\vec{f}(\hat{e}_r)$:

$$\vec{f}^{(2)}(\hat{e}_r) = -ik \int_V d^3r' \, \vec{r}' \cdot \hat{e}_r \, \vec{J}(\vec{r}') \qquad \text{(see page III.1.3)}$$

For the i-component of $\vec{f}^{(2)}$, we have

$$f_i^{(2)}(\hat{e}_r) = -\frac{ik}{r} \int_V d^3r' \, J_i(\vec{r}') \, x_j' x_j \qquad \text{(summation convention over repeated indices assumed.)}$$

$$= -\frac{ik}{2r} x_j \int_V d^3r' \Big[\underbrace{\left(J_i x_j' - J_j x_i'\right)}_{\text{M1 term}} + \underbrace{\left(J_i x_j' + J_j x_i'\right)}_{\text{E2 term: } Q_{\ell=2,m}} \Big] \triangleq f_i^{(2)\,M1} + f_i^{(2)\,E2}$$

E2 term: $Q_{\ell=2,m}$ $\longrightarrow$ we ignore it here

For the M1 term, we further get

$$f_i^{(2)\,M1} = -i\frac{k}{2r} x_j \int_V d^3r' \underbrace{\left(\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in}\right)}_{=\varepsilon_{ijk}\varepsilon_{mnk}} J_m x_n'$$

$$= -i\frac{k}{2r} \, \vec{r} \times \int_V d^3r' \left(\vec{J} \times \vec{r}'\right) \Big|_{i^{\text{th}}\text{ component}}$$

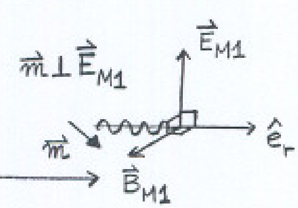
(ε_{ijk} is the fully antisymmetric Levi-Civita tensor, also known from Quantum Mechanics)

$$\leadsto \quad \underline{\vec{f}^{(2)\,M1} = ik \, \hat{e}_r \times \vec{m}} \, ,$$

where $\vec{m} = \frac{1}{2} \int_V d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')$ is the magnetic dipole (see p.(I.3.6)).

Consequently, the M1-part of the $\vec{B}$-field is

$$\boxed{\vec{B}_{M1}(\vec{r}) = -\frac{\mu_0}{4\pi} k^2 \frac{e^{ikr}}{r} \left(\hat{e}_r \times (\hat{e}_r \times \vec{m})\right)} \, .$$



Exercises: **1.** Show that

$$\boxed{\vec{E}_{M1}(\vec{r}) = -\frac{\mu_0 c}{4\pi} k^2 \frac{e^{ikr}}{r} \left(\hat{e}_r \times \vec{m}\right)}$$

2. Show that the time-average total power radiated by a magnetic dipole is

$$\boxed{\langle P_{M1} \rangle = \frac{\mu_0 \, \omega^4 m^2}{12\pi c^3}}$$

Remarks on electric and magnetic multipole radiation:

- There are the following symmetry relations:

$$\vec{B}_{M1} \xleftrightarrow{\vec{m} \leftrightarrow \vec{p}} \frac{1}{c^2} \vec{E}_{E1}$$

$$\vec{E}_{M1} \xleftrightarrow{\vec{m} \leftrightarrow \vec{p}} -\frac{1}{c^2} \vec{B}_{E1} \, .$$

- Since $\vec{p} \perp \vec{B}_{E1}$ ($\vec{p}$ lies on the plane $(\vec{E}_{E1}, \hat{e}_r)$) and $\vec{m} \perp \vec{E}_{M1}$, the polarization of light between E1 and M1 radiation differs.

- Higher-order multipole radiation is generally weaker, e.g.

$$\frac{\langle P_{E2} \rangle}{\langle P_{E1} \rangle} \sim \frac{\omega^2 Q^2_{\ell=2,m}}{100\, c^2 p^2} \sim \left(\frac{d}{\lambda}\right)^2 \; ; \qquad \omega = 2\pi \frac{c}{\lambda}$$

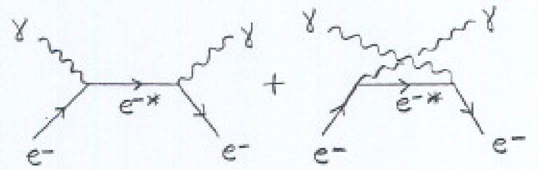
$$p \sim qd \, , \; Q_{\ell m} \sim qd^2 \, .$$

In the RZA, $d \ll \lambda$ and hence $\langle P_{E2} \rangle \ll \langle P_{E1} \rangle$.

$\langle P_{E2} \rangle$ can become dominant if the collection of charges has negligible dipole moment p.

III.3 Thomson scattering

Feynman graphs of Compton scattering:



Low-energy limit $E_{cms} \simeq m_e^2 c^4$

$\longrightarrow$ Thomson scattering (represented by the same graphs)

Consider a linearly polarized, monochromatic, plane wave hitting an electron. The polarized light is described by the electric vector

$$\vec{E} = \hat{e}\, E_0\, e^{i(\vec{k}\vec{r} - \omega t)} ,$$

where $\vec{k}$ is the so-called propagation vector (with $|\vec{k}| = \omega/c$).

The $\vec{E}$-field will exert a Lorentz force on the electron:

$$\vec{F}_L = e\vec{E} = m_e \ddot{\vec{r}} \quad \rightsquigarrow \quad \ddot{\vec{r}} = c\dot{\vec{\beta}} = \frac{e\vec{E}(t)}{m_e} .$$

Using Larmor's formula from p. (II.3.4), we find

$$\langle P_L \rangle = \frac{e^2}{6\pi\varepsilon_0 c} \langle \dot{\vec{\beta}}^2 \rangle = \frac{e^4}{6\pi\varepsilon_0 c^3} \frac{\langle \mathrm{Re}^2 \vec{E}(t) \rangle}{m_e^2} = \frac{e^4 E_0^2}{12\pi\varepsilon_0 m_e^2 c^3} .$$

The incident light carries an energy flux given by the Poynting vector

$$|\vec{S}| = \frac{1}{\mu_0} |\vec{E} \times \vec{B}| = \frac{1}{\mu_0 c} \vec{E}^2 \quad \text{and} \quad \langle |\vec{S}| \rangle = \frac{1}{2\mu_0 c} E_0^2 .$$

The scattering cross section σ_T for Thomson scattering is defined as

$$\sigma_T = \frac{\langle \text{reradiated power} \rangle}{\langle \text{incident power per unit area} \rangle} = \frac{\langle P_L \rangle}{\langle |\vec{S}| \rangle} .$$

The Thomson cross section is

$$\boxed{\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2} \right)^2 = \frac{8\pi}{3} r_e^2 = 6.652\ 10^{-29} m^2}$$

where r_e is the "classical electron radius", i.e. $r_e \approx 3.10^{-15} m$.

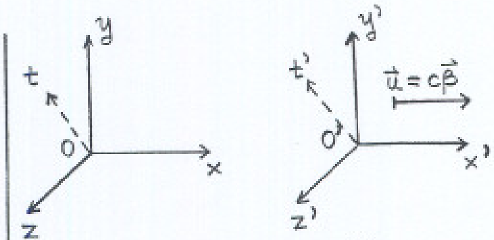
Note that σ_T is independent of frequency $\sim \omega$ in this classical low-energy limit.

Exercise: Show that $\vec{E}(t) = \hat{e}_1 E_0 e^{i(\vec{k}\vec{r} - \omega t)}$ and $\vec{B}(t) = \hat{e}_2 B_0 e^{i(\vec{k}\vec{r} - \omega t)}$, with $\hat{e}_1 \perp \hat{e}_2$, satisfy Maxwell's equations in free space.

IV. Relativistic Electromagnetism

Our aim is to show that the theory of electromagnetism is consistent with the theory of special relativity.

IV.1 Lorentz Transformations (LT)



GT: $t = t'$, $x = x' + ut'$, $y = y'$, $z = z'$

LT: $t = \gamma\left(t' + \frac{\beta}{c}x'\right)$

$x = \gamma\left(x' + \beta c t'\right)$

$y = y'$

$z = z'$; $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

→ Maxwell's equations (M1)–(M4) are not invariant under Galilean transformations (GT).

→ Eqs. (M1)–(M4) are form-invariant under LT, also derived by the two Einstein postulates:

(i) Special relativity: All laws of nature and experiments are independent of the translational motion of the system as a whole.

(ii) The existence of a universal limiting speed: There is no physical entity that can travel faster than speed of light c in vacuum at any inertial frame.

Many physical consequences; e.g. (i) Moving bodies are shorter (Fitz-Gerald contraction)

(ii) Moving clocks run slower by a factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

(iii) Newton's mechanics gets modified, e.g. $E_{tot} = \gamma m c^2$, $\vec{p}_{tot} = m\gamma\vec{u}$, etc.

⋮

Exercises: 1. Show that Newton's equation $\vec{F} = m\ddot{\vec{r}}$ is invariant under GT but not under LT.

2. Conversely, show that the Maxwell wave equation for $\vec{E}$: $\Box^2 \vec{E} \mathrel{\hat{=}} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = 0$, with $\varrho, \vec{J} = 0$, is form-invariant ($\hat{=}$ covariant) under LT but not under GT.

Fixing our notational framework:

Observe that $c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = d^2$ under LT.
So, LT preserve the modulus of a 4-dimensional vector $\underset{\sim}{x}$ (4-vector defined in a 4-dimensional vector space, the so-called Minkowski space, i.e. $\underset{\sim}{x} \cdot \underset{\sim}{x} = \underset{\sim}{x}' \cdot \underset{\sim}{x}' = d^2$ (d^2 may be of either sign!).

Two ways of representing $\underset{\sim}{x}$ and the scalar product $\underset{\sim}{x} \cdot \underset{\sim}{y}$:

(i) Euclidean representation: $\underset{\sim}{x} = (x_1, x_2, x_3, x_4)$, $\underset{\sim}{y} = (y_1, y_2, y_3, y_4)$

and $\underset{\sim}{x} \cdot \underset{\sim}{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$. However, x_4, y_4 are pure imaginary numbers.

For example, $\underset{\sim}{x} = (x, y, z, ict)$,

and hence $\underset{\sim}{x} \cdot \underset{\sim}{x} = x^2 + y^2 + z^2 - c^2t^2 = -d^2$. Heald + Marion notation.

(ii) Minkowski representation: $\underset{\sim}{x} = (x^0, x^1, x^2, x^3)$, $\underset{\sim}{y} = (y^0, y^1, y^2, y^3)$

Scalar product: $\underset{\sim}{x} * \underset{\sim}{y} = \sum_{\mu,\nu=0}^{3} x^\mu g_{\mu\nu} y^\nu$; $\mu, \nu = 0, 1, 2, 3$,

where $g_{\mu\nu} \mathrel{\hat{=}} \mathrm{diag}(1, -1, -1, -1)$ is the metric (tensor) of the Minkowski space. (What is the metric for the Euclidean space?).

So, $\underset{\sim}{x} * \underset{\sim}{x} = x^\mu g_{\mu\nu} x^\nu = d^2$ (Einstein's summation convention implied.)

where $\underset{\sim}{x} \mathrel{\hat{=}} x^\mu = (ct, x, y, z)$ is called contravariant vector.

Define now the covariant vector: $\underset{\sim}{x}^d \mathrel{\hat{=}} x_\mu = (ct, -x, -y, -z)$.

Then, the scalar product becomes (Jackson + our notation)

$\underset{\sim}{x} * \underset{\sim}{x} = \underset{\sim}{x}^d \cdot \underset{\sim}{x} = x_\mu x^\mu$, and $x_\mu = g_{\mu\nu} x^\nu$ or $x^\mu = g^{\mu\nu} x_\nu$

$g^{\mu\nu}$ raises and $g_{\mu\nu}$ lowers indice

Exercises: 1. Show that $x_\mu x^\mu = x'_\mu x'^\mu$, i.e. it is invariant under LT. How are x'^μ and x'_μ defined?

2. Show that $g_{\mu\nu} g^{\nu\lambda} = g_\mu^\lambda \mathrel{\hat{=}} \mathrm{diag}(1, 1, 1, 1) = \mathbf{1}_{4\times4}$

Rules for Lorentz algebra in Minkowski representation:

(i) Define a Lorentz vector in its contravariant 4-vector form.

E.g., the position 4-vector is defined as

$$x^\mu \triangleq (x^0, x^1, x^2, x^3) \triangleq (ct, x, y, z)$$

(ii) Calculate the covariant 4-vector by multiplying the contravariant 4-vector with the metric tensor $g_{\mu\nu}$.

E.g., the covariant position 4-vector is

$$x_\mu = g_{\mu\nu} x^\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} (x^0, x^1, x^2, x^3) = (x^0, -x^1, -x^2, -x^3).$$

Thus, $x_\mu \triangleq (x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3) \triangleq (ct, -x, -y, -z)$.

Notice that $x_\mu \neq x^\mu$, i.e. x_μ and x^μ should be seen as two different vectors related through $x_\mu = g_{\mu\nu} x^\nu$ and $x^\mu = g^{\mu\nu} x_\nu$, with $\boxed{g^{\mu\nu} \triangleq (g^{-1})_{\mu\nu} = g_{\mu\nu}}$.

(iii) Multiplication **always** occurs between a covariant and a contravariant vector or between covariant and contravariant Lorentz indices.

For two position 4-vectors x^μ and y^ν, we can form the Lorentz invariant products:

$$x^\mu g_{\mu\nu} y^\nu = x^\mu y_\mu = x_\nu y^\nu = x_\kappa g^{\kappa\lambda} y_\lambda \quad ; \quad \mu, \nu, \kappa, \lambda = 0, 1, 2, 3$$

(iv) The metric tensors $g_{\mu\nu}$ or $g^{\mu\nu}$ lower and raise Lorentz indices: $g_{\mu\nu} x^\nu = x_\mu$, but also $g_{\mu\kappa} g^{\kappa\lambda} g_{\lambda\nu} = g_{\mu\nu}$ (Why?)

Exercise: Show that for two position vectors x^μ and y^ν, the scalar product $x_\mu y^\mu$ is Lorentz invariant but not $x^\mu y^\mu$.

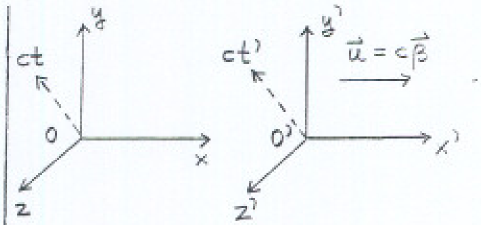
(Hint: Consider a simple example, with $\vec{x} \parallel \vec{y} \parallel \hat{e}_x$.)

(v) Minkowskian representation of the LT:

The defining equation of a LT is

$$\boxed{x'^{\mu} \triangleq \Lambda^{\mu}{}_{\nu}\, x^{\nu}} \;,$$

where $\Lambda^{\mu}{}_{\nu}$ is the 4×4 LT matrix and depends on the boost velocity $\vec{\beta}$.



LT for $\vec{\beta} \parallel \hat{e}_x \parallel \hat{e}_{x'}$:

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$
$$\boxed{\begin{array}{c} ct = x^0 \\ ct' = x'^0 \end{array}} \;;\; \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

For $\vec{\beta} \parallel \hat{e}_x$, $\Lambda^{\mu}{}_{\nu}$ reads:

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \Lambda^{\nu}{}_{\mu} \,.$$

Note that $\Lambda^{\mu}{}_{\nu} = \Lambda^{\nu}{}_{\mu}$ is a symmetric matrix, but $\Lambda^{\mu}{}_{\nu} \neq \Lambda_{\mu}{}^{\nu}$

In addition, we observe that the LT matrix is determined by

$$\boxed{\Lambda^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}}$$

Any proper contravariant 4-vector A^{μ} should transform under LT like x^{μ}:

$$A'^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$

(vi) LT of a proper covariant 4-vector:

$$x'_{\mu} = g_{\mu\kappa} x'^{\kappa} = g_{\mu\kappa} \Lambda^{\kappa}_{\lambda} x^{\lambda} = g_{\mu\kappa} \Lambda^{\kappa}_{\lambda} g^{\lambda\nu} x_{\nu} \,.$$

Since $g_{\mu\kappa} \Lambda^{\kappa}_{\lambda} g^{\lambda\nu} = (\Lambda^{-1})^{\nu}_{\mu}$, we find that $\boxed{x'_{\mu} = (\Lambda^{-1})^{\nu}_{\mu} x_{\nu}}$,

and $\boxed{(\Lambda^{-1})^{\nu}_{\mu} = \dfrac{\partial x^{\nu}}{\partial x'^{\mu}}}$; $\quad \Lambda^{\mu}_{\nu} (\Lambda^{-1})^{\nu}_{\lambda} = g^{\mu}_{\lambda} = \delta^{\mu}_{\lambda}$

Any proper covariant 4-vector B_{μ} transforms under LT as

$$B'_{\mu} = (\Lambda^{-1})^{\nu}_{\mu} B_{\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} B_{\nu} \,.$$

Exercises:

1. Show that $g_{\mu\kappa} \Lambda^{\kappa}_{\lambda} g^{\lambda\nu} = (\Lambda^{-1})^{\nu}_{\mu}$ by using the matrix representation of Λ^{μ}_{ν}, $g_{\mu\nu}$ and $(\Lambda^{-1})^{\nu}_{\mu}$ explicitly.
2. Show that the differential 4-vector $\partial_{\mu} \triangleq \frac{\partial}{\partial x^{\mu}}$ is a proper covariant vector and $\partial^{\mu} \triangleq \frac{\partial}{\partial x_{\mu}}$ is a proper contravariant one.

IV.2 Lorentz Four-Vectors

Are there more genuine Lorentz 4-vectors apart from $x^\mu = (ct, \vec{x})$? (Remember that a proper Lorentz vector A^μ satisfies: $A_\mu A^\mu = A'_\mu A'^\mu$ under an arbitrary LT, where $A'^\mu = \Lambda^\mu{}_\nu A^\nu$.)

Let us define $dx^\mu = (c\,dt, d\vec{x})$, then $dx_\mu = (c\,dt, -d\vec{x})$

and $ds^2 = dx_\mu dx^\mu = c^2dt^2 - (d\vec{x})^2$ is a Lorentz-invariant quantity.

Likewise, the quantity

$$d\tau \triangleq \frac{1}{c}\, ds = \frac{1}{c}\sqrt{dx_\mu dx^\mu} = \sqrt{dt^2 - \frac{1}{c^2}(d\vec{x})^2} = dt\sqrt{1 - \frac{1}{c^2}\underbrace{\left(\frac{d\vec{x}}{dt}\right)^2}_{\vec{u}^2}}$$

$$= dt\sqrt{1-\beta^2} = dt/\gamma$$

is Lorentz-invariant as well.

The quantity $d\tau$ is called the proper time. It is the time interval measured in a moving frame O' by a stationary clock attached to O', i.e. $d\tau$ is Lorentz invariant due to Einstein's first postulate of special relativity.

With the help of $d\tau$, we can construct the 4-vector velocity u^μ:

$$u^\mu \triangleq \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \underbrace{\frac{d\vec{x}}{d\tau}}_{:\frac{dt}{d\tau}\frac{d\vec{x}}{dt}}\right) = (c\gamma, \gamma\vec{u})$$

By analogy, the 4-vector momentum is defined as

$$p^\mu \triangleq m u^\mu = (mc\gamma, m\gamma\vec{u}) = \left(\frac{E}{c}, \vec{p}\right),$$

where E is the energy of the particle and $\vec{p}$ its momentum.

Since $p_\mu = g_{\mu\nu}p^\nu = \left(\frac{E}{c}, -\vec{p}\right)$, we easily verify that

$$p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p}^2 = \frac{1}{c^2}\left(E^2 - \vec{p}^2c^2\right) = m^2c^2,$$

which is clearly Lorentz invariant, as it should be for a true 4-vector.

Four-vectors in Electrodynamics:

Let us now see whether other 4-vectors related to Electrodynamics exist.

Consider the continuity equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{J} = 0,$$

which has to hold for any inertial frame. Having defined the 4-vector current density as (contravariant form)

$$J^\mu = (c\rho, \vec{J}),$$

we can write the continuity equation as

$$\frac{1}{c}\frac{\partial}{\partial t}J^0 + \frac{\partial}{\partial x^i}\underbrace{\hat{e}_i\vec{J}}_{J^i} = 0 \rightsquigarrow \underline{\frac{\partial}{\partial x^\mu}J^\mu = 0}$$

or even shorter as $\underline{\partial_\mu J^\mu = 0}$, with $\partial_\mu \mathrel{\hat{=}} \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

Note that $\partial_\mu \mathrel{\hat{=}} \frac{\partial}{\partial x^\mu}$ transforms as a covariant vector. (see exercise 2 on p. IV.1.4).

In the Lorentz gauge, the Maxwell equations for Φ and $\vec{A}$ read:

see p. II.1.1

$$\begin{cases} \frac{1}{c^2}\frac{\partial}{\partial t^2}\Phi - \vec{\nabla}^2\Phi = \frac{\rho}{\varepsilon_0} = \mu_0 c^2 \rho \\ \frac{1}{c^2}\frac{\partial}{\partial t^2}\vec{A} - \vec{\nabla}^2\vec{A} = \mu_0\vec{J} \end{cases}$$

with the Lorentz gauge:
$$\frac{1}{c^2}\frac{\partial}{\partial t}\Phi + \vec{\nabla}\vec{A} = 0$$

The above equations suggest to use the 4-vector potential:

$$A^\mu \mathrel{\hat{=}} \left(\frac{\Phi}{c}, \vec{A}\right).$$

Then, the Lorentz gauge can be written in a manifestly Lorentz invariant form:

$$\frac{\partial}{\partial x^\mu}A^\mu = 0 \quad \text{or} \quad \partial_\mu A^\mu = 0.$$

Exercise: If ρ_0 is the rest charge density in a static frame with $\vec{J}=0$, show that in a moving frame with velocity $\vec{u}$, the 4-vector current density is $J^\mu = \rho_0 u^\mu = \rho_0(c\gamma, \gamma\vec{u})$

Finally, the Maxwell equations are written in the covariant form:

$$\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} A^\mu = \mu_0 J^\mu \quad \text{or} \quad \partial^\nu \partial_\nu A^\mu = \mu_0 J^\mu \quad \text{or} \quad \Box^2 A^\mu = \mu_0 J^\mu ,$$

where $\Box^2 \triangleq \partial_\nu \partial^\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is called D'Alembertian operator and is Lorentz invariant (Why?).

$\longleftrightarrow$

Exercises: 1. Show explicitly that the D'Alembertian operator remains invariant under a LT.

2. Show that the Maxwell equation for the 4-vector potential without assuming any specific gauge condition is given by

$$\left(g_{\mu\nu}\Box^2 - \partial_\mu \partial_\nu \right) A^\nu = \mu_0 J_\mu .$$

Comment: There is no Green function for the operator $g_{\mu\nu}\Box^2 - \partial_\mu\partial_\nu$ and choosing the gauge is a necessity in Field Theory.

IV.3 Electromagnetic Field Tensor

Question: How do $\vec{E}$ and $\vec{B}$ fields change under a LT?

They cannot be 4-vectors, but what are they?

We know that $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$. We try to write these 6 equations in terms of $\partial^\mu = \frac{\partial}{\partial x_\mu}$ and A^μ as follows:

$\vec{E}$-field:

$$\frac{E_i}{c} = -\frac{\partial A^0}{\partial x^i} - \frac{\partial A^i}{\partial x^0} = \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0}\,, \quad \text{since } A^\mu = \left(\frac{\Phi}{c}, \vec{A}\right),$$

$$\rightsquigarrow -\frac{E_i}{c} = \frac{\partial A^i}{\partial x_0} - \frac{\partial A^0}{\partial x_i} \triangleq F^{0i}$$

and $x^\mu = (x^0, \vec{x})$ and $x_\mu = (x^0, -\vec{x})$.

$\vec{B}$-field:

$$B_1 = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A^2}{\partial x_3} - \frac{\partial A^3}{\partial x_2} = F^{32}.$$

Similarly, $B_2 = -\left(\frac{\partial A^3}{\partial x^1} - \frac{\partial A^1}{\partial x^3}\right) = \frac{\partial A^3}{\partial x_1} - \frac{\partial A^1}{\partial x_3} = F^{13}$

and $B_3 = \frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2} = \frac{\partial A^1}{\partial x_2} - \frac{\partial A^2}{\partial x_1} = F^{21}$.

So, putting everything together, we have

$$F^{\mu\nu} \triangleq \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{bmatrix} = -F^{\nu\mu}.$$

E_i and B_i are the 6 elements of the antisymmetric 4x4 matrix $F^{\mu\nu}$. $F^{\mu\nu}$ is called electromagnetic field tensor, as it tranforms as a tensor of rank 2 under a LT:

$$F'^{\mu\nu} = \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta} F^{\alpha\beta}, \quad \text{with } F'^{\mu\nu} = \frac{\partial A'^\nu}{\partial x'_\mu} - \frac{\partial A'^\mu}{\partial x'_\nu} = \partial'^\mu A'^\nu - \partial'^\nu A'^\mu.$$

Exercises: **1.** Show that $F^{\mu\nu}$ is a rank 2 Lorentz tensor.

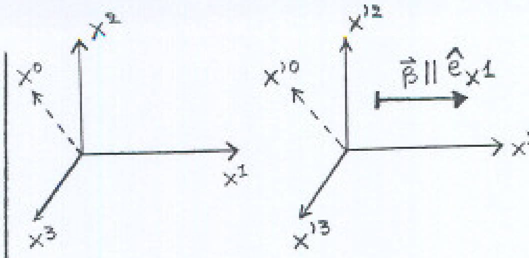
2. Show that (M2) and (M3) may be expressed by

$$\boxed{\partial^\mu F^{\nu\lambda} + \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} = 0.},$$

and (M1) and (M4) by

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}$$

LT of $\vec{E}$ and $\vec{B}$ fields:



$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$$

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Consider the specific LT, corresponding to a boost along the $x^1 \equiv x$ axis.

Then, $F^{\mu\nu}$ transforms as

$$F'^{\mu\nu} = \Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} F^{\alpha\beta}$$

or in matrix form

$$F' = \Lambda F \Lambda .$$

So,

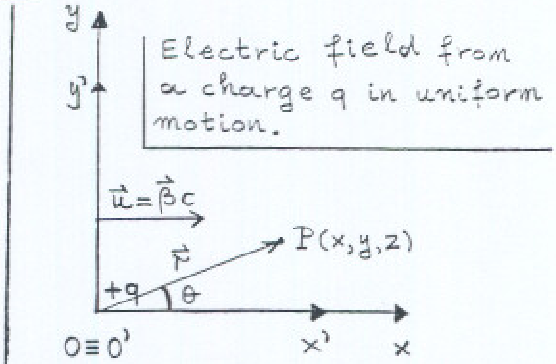
$$F'^{\mu\nu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\overbrace{\begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{bmatrix}}^{:F^{\alpha\beta}} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\begin{bmatrix} \frac{1}{c}\gamma\beta E_1 & -\frac{1}{c}\gamma E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 \\ \frac{1}{c}\gamma E_1 & -\frac{1}{c}\gamma\beta E_1 & -B_3 & B_2 \\ \frac{1}{c}\gamma E_2 - \gamma\beta B_3 & -\frac{1}{c}\gamma\beta E_2 + \gamma B_3 & 0 & -B_1 \\ \frac{1}{c}\gamma E_3 + \gamma\beta B_2 & -\frac{1}{c}\gamma\beta E_3 - \gamma B_2 & B_1 & 0 \end{bmatrix}}$$

$$F'^{\mu\nu} = \begin{bmatrix} 0 & -\frac{1}{c}E_1 & -\gamma\left(\frac{1}{c}E_2 - \beta B_3\right) & -\gamma\left(\frac{1}{c}E_3 + \beta B_2\right) \\ \frac{1}{c}E_1 & 0 & -\gamma\left(B_3 - \frac{\beta}{c}E_2\right) & \gamma\left(B_2 + \frac{\beta}{c}E_3\right) \\ \gamma\left(\frac{1}{c}E_2 - \beta B_3\right) & \gamma\left(B_3 - \frac{\beta}{c}E_2\right) & 0 & -B_1 \\ \gamma\left(\frac{1}{c}E_3 + \beta B_2\right) & -\gamma\left(B_2 + \frac{\beta}{c}E_3\right) & B_1 & 0 \end{bmatrix}$$

Consequently,

$$\boxed{\begin{array}{ll} E'_1 = E_1 & B'_1 = B_1 \\ E'_2 = \gamma\left(E_2 - \beta c B_3\right) & B'_2 = \gamma\left(B_2 + \frac{\beta}{c}E_3\right) \\ E'_3 = \gamma\left(E_3 + \beta c B_2\right) & B'_3 = \gamma\left(B_3 - \frac{\beta}{c}E_2\right) \end{array}}$$

Liénard - Wiechert electric field from LT:



Charge q situated at the origin of the moving frame. Stationary and moving frame overlap at $t=t'=0$.

A charge q is at rest of the moving frame O'.

At $t=t'=0$, both the stationary frame O and the moving frame O' overlap.

At O' (for all t'), the $\vec{E}$-field takes on the known form

$$\vec{E}' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}\vec{r}' \quad \text{and } \vec{B}'=0 .$$

At $t=0$, the $\vec{E}$-field in the frame O is computed by

$$\left.\begin{array}{l|l} E_1' = E_1 & B_1' = B_1 = 0 \\ E_2' = \gamma(E_2 - \beta c B_3) & B_2' = \gamma\left(B_2 + \frac{\beta}{c}E_3\right) = 0 \\ E_3' = \gamma(E_3 + \beta c B_2) & B_3' = \gamma\left(B_3 - \frac{\beta}{c}E_2\right) = 0 \end{array}\right\} \Longrightarrow \left|\begin{array}{l} E_1 = E_1' \\ E_2 = \gamma E_2' \\ E_3 = \gamma E_3' \end{array}\right.$$

Thus, $\vec{E} = (E_1', \gamma E_2', \gamma E_3') = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}(x', \gamma y', \gamma z')$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^3}\gamma(x,y,z)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q\gamma}{r'^3}\vec{r}$$

$$\boxed{\begin{array}{l} x' = \gamma(x-ut) \\ \quad = \gamma x, \ \text{at } t=0 \\ y'=y, \ z'=z \end{array}}$$

Note that E-field lines point to the present position.

On the other hand, $r'^2 = x'^2 + y'^2 + z'^2 = \gamma^2x^2 + y^2 + z^2$

$$= \gamma^2 r^2\cos^2\theta + r^2\sin^2\theta = \gamma^2 r^2\Big[\cos^2\theta + \underbrace{(1-\beta^2)}_{1/\gamma^2}\sin^2\theta\Big]$$
$$= \gamma^2 r^2\left[1-\beta^2\sin^2\theta\right].$$

Putting everything together, we get

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\gamma\vec{r}}{\gamma^3 r^3(1-\beta^2\sin^2\theta)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{q(1-\beta^2)\vec{r}}{r^3(1-\beta^2\sin^2\theta)^{3/2}} .$$

This is exactly the Liénard - Wiechert formula for the $\vec{E}$-field of a uniformly moving charge point! (see p. II.3.2)