Lectures on Gravitation

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7. Friedmann–Robertson–Walker (FRW) Metric

- Expansion, Isotropy and Homogeneity
- FRW Metric
- Friedmann and Raychaudhuri Equations
- Matter-, Radiation- and Vacuum Energy- Dominated Universes
- Cosmological Redschift
- Cosmological Distance Measures
- The Flatness Problem

1. Preliminaries

1. Electromagnetism → Quantum ElectroDynamics:
U$(1)_{\rm em}$
Force carrier: photon, γ, massless, spin $= 1\hbar$
Coupling to charged matter particles, such as e, u, d quarks.
Strength of the coupling $\alpha_{\rm em}(m_e) = 1/137$.

2. Weak interactions → Quantum WeakDynamics:
SU$(2)_L\otimes$U$(1)_Y/$U$(1)_{\rm em}$
Force carriers: W^+, W^-, Z bosons, massive, spin $= 1\hbar$
Coupling to particles with weak charges.
Strength of the coupling $\alpha_w(M_Z) \approx 1/30$.
Observed weakness due to the massiveness of $W^\pm$ and Z: $M_W, M_Z \sim 100$ GeV.

3. Strong interactions → Quantum ChromoDynamics:
SU$(3)_{\rm color}$
Force carriers: 8 massless gluons, g^a, spin $= 1\hbar$
Coupling to coloured particles, such as u, d quarks.
Strength of the coupling $\alpha_s(M_Z) \approx 1/10$.

4. Gravity → Quantum Gravity (?):
No known self-consistent quantum theory: Superstrings, large groups (E_6, etc.), extra dims. (?).
Force carrier: massless gravitons, with spin $= 2\hbar$.
$\sim 10^{-40}$ weaker than Electromagnetism.

– Literature

Recommended Texts:

- R. D'Inverno, *Introducing Einstein's Relativity*, Oxford University Press
  Chapters: 2,3,4,8 (SR); 5,6,9,10,12,13,14,15,16,20,22,23 (GR).
- J.B. Hartle, *An Introduction to Einstein's General Relativity*, Addison Wesley.
- D. McMahon, *Relativity Demystified*, McGraw Hill.

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Advanced Texts:

- S. Weinberg, *Gravitation and Cosmology*, Wiley.
- C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, Freeman
- R.M. Wald, *General Relativity*, University of Chicago Press
- L.H. Ryder, *Quantum Field Theory*, Cambridge University Press.

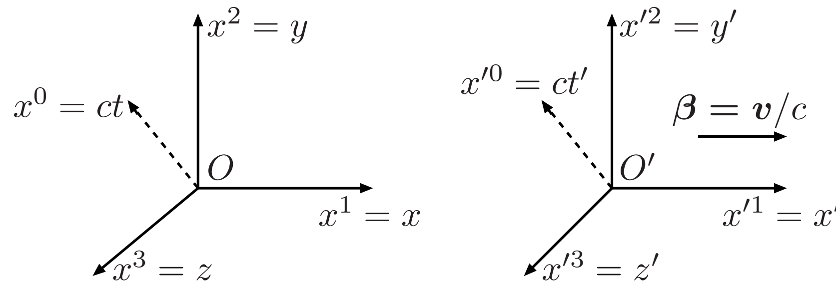
– Principles of Special Relativity (SR)

Einstein's postulates for SR:

(i) All laws of nature are the same for all inertial observers.

(ii) The speed of light c is the same in all inertial systems.

Lorentz transformations (LT): Space and time are not absolute but related by LT



$$ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z,$$
$$\Longrightarrow \quad c^2t'^2 - x'^2 - y'^2 - z'^2 \ = \ c^2t^2 - x^2 - y^2 - z^2$$

Physical consequences:
– Moving bodies are shorter by a factor $\gamma = 1/\sqrt{1-\beta^2}$
– Moving clocks run slower by a factor γ
– Newton's mechanics gets modified: $E = \gamma mc^2$, $\mathbf{p} = m\gamma\mathbf{v}$
– Maxwell's equations are consistent with SR
– . . .

– Covariant Formulation of Special Relativity

Rules for Lorentz algebra:

(i) Define the *contravariant* position 4-vector x^μ:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

(ii) Find the *covariant* position 4-vector with the use of the *flat* or *Minkowski* metric $\boldsymbol{\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)}$:

$$x_\mu = \eta_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3)$$

(iii) Lower and raise Lorentz indices with $\eta^{\mu\nu} \equiv (\eta^{-1})_{\mu\nu} = \eta_{\mu\nu}$ and $\eta_{\mu\nu}$:

$$x^\mu = \eta^{\mu\nu} x_\nu\,, \qquad x_\mu = \eta_{\mu\nu} x^\nu$$

(iv) Sum or contract *always* covariant with contravariant Lorentz indices:

$$x^\mu \eta_{\mu\nu} y^\nu = x_\nu y^\nu = x^\mu y_\mu$$

Exercise: Show that $\eta^{\mu\nu}\eta_{\nu\lambda} = \eta^\mu{}_\lambda = \delta^\mu{}_\lambda = \mathbf{1}_4$.

(v) LT of the contravariant position 4-vector:

$$x'^\mu = \Lambda^\mu{}_\nu(\boldsymbol{\beta})\, x^\nu\,,$$

where $x^\mu = (ct, x, y, z)$, $x'^\mu = (ct', x', y', z')$ are the contravariant position 4-vectors in O and O' frames, and

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{for } \boldsymbol{\beta} \parallel \mathbf{e}_x\,.$$

Under a Lorentz trans, we have $x^\mu x_\mu = x'^\mu x'_\mu$ or

$$x^\mu \eta_{\mu\nu} x^\nu = x^\beta \Lambda^\mu{}_\beta \eta_{\mu\nu} \Lambda^\nu{}_\alpha x^\alpha \;\Rightarrow\; \Lambda^T \eta \Lambda = \eta\,,$$

$\therefore$ $\Lambda^\mu{}_\nu \in$ SO(1,3), with $\det\Lambda = 1$.

LT of a proper *contravariant* 4-vector A^μ: $A'^\mu = \Lambda^\mu{}_\nu A^\nu$.

(vi) LT of a proper *covariant* 4-vector B_μ: $B'_\mu = (\Lambda^{-1})^\nu{}_\mu B_\nu$.

Exercises:

1. Show that $x'_\mu = \eta_{\mu\kappa}\Lambda^\kappa{}_\lambda \eta^{\lambda\nu}\, x_\nu$ and

$$\eta_{\mu\kappa}\Lambda^\kappa{}_\lambda \eta^{\lambda\nu} = \Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu\,.$$

2. Derive the useful relations:

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \frac{\partial x_\nu}{\partial x'_\mu}\,, \qquad (\Lambda^{-1})^\nu{}_\mu = \frac{\partial x'_\mu}{\partial x_\nu} = \frac{\partial x^\nu}{\partial x'^\mu}\,.$$

Displacement position 4-vector:

$$dx^\mu \;=\; (cdt\,,\, d\mathbf{x})\,, \quad dx_\mu = \eta_{\mu\lambda}\, dx^\lambda = (cdt\,,\, -d\mathbf{x})\;.$$

Lorentz invariant line element:

$$ds^2 \;=\; dx^\kappa dx_\kappa \;=\; c^2 dt^2 - (d\mathbf{x})^2$$

Proper time:

$$d\tau \;=\; \frac{1}{c}\sqrt{dx^\nu dx_\nu} \;=\; \frac{dt}{\gamma}$$

Contravariant 4-velocity u^μ and 4-momentum p^μ:

$$u^\mu \;=\; \frac{dx^\mu}{d\tau} \;=\; (c\gamma\,,\,\gamma\mathbf{v})\,, \qquad p^\mu \;=\; mu^\mu \;=\; (E/c\,,\,\mathbf{p})$$

Charge conservation in relativistic Electrodynamics:

$$\frac{\partial\rho}{dt} \;+\; \nabla\cdot\mathbf{J} \;=\; 0 \quad\Rightarrow\quad \partial_\mu J^\mu \;=\; 0; \quad J^\mu = (c\rho, \mathbf{J})$$

The Lorentz gauge in relativistic Electrodynamics:

$$\frac{\partial\Phi}{c^2\, dt} \;+\; \nabla\cdot\mathbf{A} \;=\; 0 \quad\Rightarrow\quad \partial_\mu A^\mu \;=\; 0; \quad A^\mu = (\Phi/c, \mathbf{A})$$

Exercise: Show that $\partial_\mu \;\equiv\; \frac{\partial}{\partial x^\mu} \;=\; \left(\frac{\partial}{c\partial t}\,,\,\nabla\right)$ transforms as a proper *covariant* 4-vector: $\partial'_\mu \;=\; (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu$.

– Einstein's Lift Experiments and Gravitational Redshift

Einstein's lift experiments revealed the following 2 principles:

Weak Equivalence Principle (WEP): The gravitational field couples in the same *universal* way to all mass and energy.

Strong Equivalence Principle (SEP): The laws of physics are the same in an accelerated frame and in an uniform and static gravitational field.

Consequences:

- Inertial and gravitational masses are equal. Eötvös experiment in 1889 found agreement to 1 part in 10^5, which improved a 100 years later to 1 part in 10^{13}.
- Free-falling observers in a gravitional field do not experience gravity *locally*.

Exercise****:* Does a free-falling electrically charged particle radiate and why?

The Doppler-shift Effect:

$$\frac{\lambda}{\lambda_0} \;=\; \gamma\,(1 + v_r/c)\,,$$

λ: wavelength observed
λ_0: wavelength emitted in source's rest frame
v_r: radial component of the velocity $\mathbf{v}$ of a point-like source moving away from the observer.

For $v_r = v = |\mathbf{v}|$, the Doppler-shift formula reduces to

$$\frac{\lambda}{\lambda_0} \;=\; \left(\frac{1+v/c}{1-v/c}\right)^{1/2} \;=\; 1 \,+\, v/c \,+\, \mathcal{O}(v^2/c^2)$$

Gravitational Redshift:

Photons falling freely in a gravitational field $\Phi(r)$ experience a change of frequency $\Delta\nu$ given by

$$\frac{\Delta\nu}{\nu} \;=\; -\,\frac{1}{c^2}\,\Delta\Phi\,,$$

where $\Phi(r) = -GM/r$ is the gravitational potential outside of a spherical body of mass M.

$\therefore$ Photons climbing up a gravitational potential get redshifted.

Einstein's Vision of General Relativity:

The following 3 points played an essential role in Einstein's formulation of GR:

(i) Spacetime is a 4-dimensional Riemannian manifold (to be discussed in Section 2) endowed with a position-dependent metric:
$$ds^2 \;=\; g_{\mu\nu}(x^\rho)\,dx^\mu\,dx^\nu\;.$$
According to the Equivalence Principle, one can always choose coordinates, such that the space is *locally* flat, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$ and the geodesics are straight lines.

(ii) The geodesics can be classified as null ($ds^2 = 0$), time-like ($ds^2 > 0$) and space-like ($ds^2 < 0$). Light rays follow null geodesics, whereas free-falling massive particles move along time-like geodesics.

(iii) Any form of energy and momentum curves spacetime.

Exercise: A lift makes a free-fall in a homogeneous gravitational field. Use the SEP to derive the trajectory of a photon emitted horizontally within the lift as viewed from an observer on the ground.

2. Manifolds, Metrics and Tensors

– Manifolds, Curves and Surfaces

Manifold: A continuous space of points which can be described locally by n-dimensional Euclidean or Minkowskian geometry.

The manifold may not be covered by a single coordinate system, so a set of different overlapping coordinate systems needs be introduced. The complete set of coordinate systems is called **Atlas**.

Curves: A curve in an n-dimensional manifold or n-manifold is a subset of points defined *parametrically* by the equations

$$x^a = x^a(\lambda); \quad a = 1, 2, \ldots, n,$$

where x^a is the n-dimensional position vector and λ is an arbitrary real parameter.

Surfaces: An m-dim hypersurface in an n-manifold $(m < n)$ is a subset of points defined *parametrically* by

$$x^a = x^a(\lambda_1, \lambda_2, \ldots, \lambda_m) ,$$

where $\lambda_{1,2,\ldots,m}$ are arbitrary real parameters.

Exercise: Write down the parametric equations that define a circle and the surface of a sphere (without the poles) in a 3-dim Euclidean space.

– Coordinate Transformations and Tangent Vectors

Coordinate transformations: In a given point of a manifold, a coordinate transformation from $x^\mu \to x'^\mu = x'^\mu(x^\nu)$ is defined by means of the contravariant displacement position vectors dx^μ and dx'^μ:

$$dx'^\mu = J^\mu_{\ \nu}\, dx^\nu , \qquad dx^\mu = (J^{-1})^\mu_{\ \nu}\, dx'^\nu ,$$

where $J^\mu_{\ \nu}$ and $(J^{-1})^\mu_{\ \nu}$ are in general position dependent. They are given by

$$J^\mu_{\ \nu} = \frac{\partial x'^\mu}{\partial x^\nu} , \qquad (J^{-1})^\mu_{\ \nu} = \frac{\partial x^\mu}{\partial x'^\nu}$$

Notice that $J^\mu_{\ \nu}\,(J^{-1})^\nu_{\ \rho} = \delta^\mu_{\ \rho}$.

Tangent Vectors: For a curve parameterized as $x^\mu(u)$, the tangent vector T^μ is given by

$$T^\mu = \frac{dx^\mu}{du} .$$

T^μ transforms as the infinitesimal displacement vector dx^μ:

$$T'^\mu = J^\mu_{\ \nu}\, T^\nu .$$

Exercise: Find the parametric representation of the curve $y = x^2$ in a 2-dimensional Euclidean space, such that $x \geq 1$. From this, compute the tangent vector on this curve.

– Metric and Line Element

To measure distances on a manifold, we need to define a **metric** $g_{\mu\nu}$ via the square of the **line element** ds:

$$ds^2 \;=\; g_{\mu\nu}\, dx^\mu\, dx^\nu$$

Since the line element does not change its value under a coordinate transformation, this implies that

$$g'_{\alpha\beta} \;=\; (J^{-1})^\mu{}_\alpha\, (J^{-1})^\nu{}_\beta\, g_{\mu\nu}$$

is the metric in the transformed coordinate system. The inverse of the metric $g_{\mu\nu}$ is denoted as $g^{\mu\nu}$ and is determined by the relation: $g^{\mu\nu}\, g_{\nu\rho} \;=\; \delta^\mu{}_\rho$.

A manifold endowed with a metric is called **Riemannian manifold**.

The metric of a Riemannian manifold is in general position dependent, i.e. $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$.

Exercise: Write down the metrics and line elements for plane polar, spherical polar and cylindrical polar coordinates.

– 4-Vectors and Tensors

A contravariant **vector** or simply vector A^μ is an array defined at a given point x^μ of the manifold which transforms, under a coordinate transformation $x^\mu \to x'^\mu$, like the tangent vector T^μ at the same point:

$$A'^\mu \;=\; J^\mu{}_\nu\, A^\nu\,, \qquad A^\mu \;=\; (J^{-1})^\mu{}_\nu\, A'^\nu$$

A **co-vector** A_μ transforms by the inverse of $J^\mu{}_\nu$:

$$A'_\mu \;=\; (J^{-1})^\nu{}_\mu\, A_\nu\,, \qquad A_\mu \;=\; J^\nu{}_\mu\, A'_\nu\,.$$

Alternatively, a co-vector may be defined as $A_\mu = g_{\mu\nu} A^\nu$. Evidently, it is $A^\mu = g^{\mu\nu}\, A_\nu$.

Exercise: Show that a co-vector defined alternatively via the metric $g_{\mu\nu}$, $A_\mu = g_{\mu\nu} A^\nu$ has the proper transformation properties.

Tensors: An $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor transforms like

$$A'^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} \;=\; J^{\mu_1}{}_{\alpha_1} \ldots J^{\mu_p}{}_{\alpha_p}\, (J^{-1})^{\beta_1}{}_{\nu_1} \ldots (J^{-1})^{\beta_q}{}_{\nu_q}\, A^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q}$$

Exercise: Write down the transformation properties of an $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

– Conformal Metrics

Inner product: The inner or scalar product between two vectors is defined by

$$\langle S, T \rangle \;=\; S_\mu T^\mu \;=\; S^\mu T_\mu \;=\; g_{\mu\nu}\, S^\mu\, T^\nu \;.$$

Note that the scalar product between 2 vectors is invariant under coordinate transformations.

The modulus of a vector S^μ is defined as

$$||S|| \;=\; \begin{cases} (S^\mu S_\mu)^{1/2}, & \text{for time-like vectors} \\ (-S^\mu S_\mu)^{1/2}, & \text{for space-like vectors} \end{cases} \;.$$

Relative angle between two vectors S^μ and T^μ is given by

$$\cos\theta \;=\; \frac{\langle S, T \rangle}{||S||\; ||T||} \;.$$

Conformal metrics: Coordinate transformations that maintain the angle between 2 vectors are called conformal. The correspoding metrics associated to those coordinate transformations are called conformal metrics.

If $g_{\mu\nu}$ is given, a set of conformal metrics is obtained by

$$\widetilde{g}_{\mu\nu} \;=\; \Omega(x^\rho)\; g_{\mu\nu} \;,$$

where Ω is an arbitrary non-vanishing function, i.e. $\Omega(x^\rho) \neq 0$.

3. Connection and Tensor Calculus

– Covariant Differentiation and Parallel Transport

Partial differentiation of a vector is not a proper tensor (*Why?*):

$$\partial'_\mu A'^\nu \;=\; (J^{-1})^\alpha{}_\mu\, J^\nu{}_\beta\, \partial_\alpha A^\beta \;+\; (J^{-1})^\alpha{}_\mu\, A^\beta\, (\partial_\alpha J^\nu{}_\beta)$$

Define a vector (along with its basis vectors) as

$$\mathbf{A}(x) \;=\; A^\nu(x)\; \mathbf{e}_\nu(x)$$

NB: $A'^\mu = J^\mu{}_\alpha A^\alpha$, $\mathbf{e}'_\mu = (J^{-1})^\beta{}_\mu\, \mathbf{e}_\beta \quad \Rightarrow \quad \mathbf{A} = \mathbf{A}'$.

Partial differentiaton of $\mathbf{A}$:

$$\partial_\mu \mathbf{A} \;=\; (\partial_\mu A^\nu)\, \mathbf{e}_\nu \;+\; A^\nu\, (\partial_\mu \mathbf{e}_\nu)$$

Affine connection $\Gamma^\rho{}_{\mu\nu}$**:**

$$\partial_\mu \mathbf{e}_\nu \;=\; \Gamma^\rho{}_{\mu\nu}\; \mathbf{e}_\rho$$

Substituting this into $\partial_\mu \mathbf{A}$ gives

$$\partial_\mu \mathbf{A} \;=\; \mathbf{e}_\nu\; \left(\partial_\mu A^\nu \;+\; \Gamma^\nu{}_{\mu\beta}\; A^\beta\right)$$

Covariant differentiation of a vector A^ν**:**

$$\nabla_\mu A^\nu \;=\; \partial_\mu A^\nu \;+\; \Gamma^\nu{}_{\mu\beta}\; A^\beta$$

Covariant differentiation of a scalar function ϕ:

$$\nabla_\mu \phi \;=\; \partial_\mu \phi$$

Covariant differentiation of a co-vector A_ν:

$$\nabla_\mu A_\nu \;=\; \partial_\mu A_\nu \;-\; \Gamma^\alpha_{\ \ \mu\nu}\, A_\alpha$$

Covariant differentiation of tensors:

$$\begin{aligned}
\nabla_\rho A^{\mu\nu} &= \partial_\rho A^{\mu\nu} \;+\; \Gamma^\mu_{\ \ \rho\alpha} A^{\alpha\nu} \;+\; \Gamma^\nu_{\ \ \rho\beta} A^{\mu\beta} \\
\nabla_\rho A_{\mu\nu} &= \partial_\rho A_{\mu\nu} \;-\; \Gamma^\alpha_{\ \ \rho\mu} A_{\alpha\nu} \;-\; \Gamma^\beta_{\ \ \rho\nu} A_{\mu\beta} \\
\nabla_\rho A^\mu_{\ \ \nu} &= \partial_\rho A^\mu_{\ \ \nu} \;+\; \Gamma^\mu_{\ \ \rho\alpha} A^\alpha_{\ \ \nu} \;-\; \Gamma^\beta_{\ \ \rho\nu} A^\mu_{\ \ \beta} \\
\nabla_\rho A^{\mu_1\ldots\mu_p}_{\qquad \nu_1\ldots\nu_q} &= \partial_\rho A^{\mu_1\ldots\mu_p}_{\qquad \nu_1\ldots\nu_q} \\
&\quad + \sum_{r=1}^{p} \Gamma^{\mu_r}_{\ \ \rho\alpha_r}\, A^{\mu_1\ldots\mu_{r-1}\alpha_r\mu_{r+1}\ldots\mu_p}_{\qquad \nu_1\ldots\nu_q} \\
&\quad - \sum_{r=1}^{q} \Gamma^{\beta_r}_{\ \ \rho\nu_r}\, A^{\mu_1\ldots\mu_p}_{\qquad \nu_1\ldots\nu_{r-1}\beta_r\nu_{r+1}\ldots\nu_q}
\end{aligned}$$

Exercise: Use the relations of the covariant differentiation for the vector A^ν and its co-vector A_ν to show that $\nabla_\mu(A^\nu A_\nu) = \partial_\mu(A^\nu A_\nu)$.

Parallel transport: Covariant (or tensorial) differentiation of a vector A^μ may also be introduced by means of **parallel transport**. This is an operation that allows to compare a vector A^μ at 2 different nearby points of a manifold, e.g. $A^\mu(x^\nu)$ and $A^\mu(x^\nu + \delta x^\nu)$,

According to this operation, a vector A^μ is moved from x^μ to $x'^\mu = x^\mu + \delta x^\mu$ by keeping its angle fixed with the local basis vectors $\mathbf{e}_\nu(x)$.

The change of the vector caused by parallel transport over a small interval δx^ν is given by

$$\bar{\delta} A^\mu \;=\; -\,\Gamma^\mu_{\ \ \nu\lambda}\,\delta x^\nu\, A^\lambda$$

The difference of the vector at x'^μ, $A^\mu(x')$, with the $A^\mu(x)$ after parallel transportation from x^μ to $x'^\mu = x^\mu + \delta x^\mu$, $A^\mu(x)+\bar{\delta}A^\mu$, can be used to define a covariant differentiation:

$$\begin{aligned}
DA^\mu &= A^\mu(x') \;-\; \left(A^\mu(x) + \bar{\delta}A^\mu\right) \\
&= \left(\partial_\nu A^\mu + \Gamma^\mu_{\ \ \nu\lambda}\, A^\lambda\right)\delta x^\nu \\
&= \left(\nabla_\nu A^\mu\right)\delta x^\nu
\end{aligned}$$

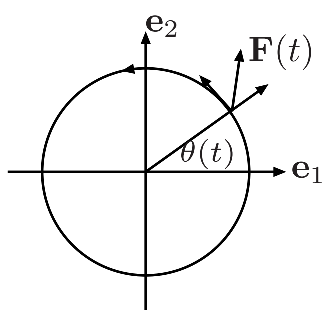
Exercise: Show that $\nabla_\mu\, \mathbf{e}_\nu \;=\; 0$.

Absolute derivative: Consider a curve parameterized by $x^\mu = x^\mu(u)$ and a vector A^μ acting on this, i.e. $A^\mu(u) = A^\mu[x^\nu(u)]$. As above, one may show that the derivative dA^μ/du is not a proper vector. Instead, the proper **absolute derivative** DA^μ/Du may be found by means of covariant differentiation:

$$\begin{aligned}\frac{DA^\mu}{Du} &= \frac{dx^\nu}{du}\,\nabla_\nu A^\mu = \frac{dx^\nu}{du}\left(\partial_\nu A^\mu + \Gamma^\mu{}_{\nu\beta}\,A^\beta\right)\\ &= \frac{dA^\mu}{du} + \Gamma^\mu{}_{\nu\beta}\,T^\nu\,A^\beta\,,\end{aligned}$$

where $T^\nu = dx^\nu/du$ is the tangent vector on the curve $x^\nu = x^\nu(u)$ (see p. 14).

<u>Exercise</u>: *A simple 2-dimensional example*



Show that the absolute time derivative is given by

$$D_t F_i(t) = \partial_t F_i(t) + (\boldsymbol{\omega} \times \mathbf{F}(t))_i\,,$$

with $\omega = \dot\theta(t)$, as known from Classical Mechanics between rotating and fixed frames. What is the affine connection?

– Affine Connection and Torsion

Affine connection $\Gamma'^\rho{}_{\mu\nu}$ may be defined via a covariant differentiation (see p. 18) or parallel transport. Under a coord. trans., it is

$$\Gamma'^\rho{}_{\mu\nu} = J^\rho{}_\gamma (J^{-1})^\alpha{}_\mu (J^{-1})^\beta{}_\nu\,\Gamma^\gamma{}_{\alpha\beta} + J^\rho{}_\gamma (J^{-1})^\alpha{}_\mu \partial_\alpha (J^{-1})^\gamma{}_\nu$$

Alternatively, a connection that transforms as above is called affine connection.

Torsion $T^\rho{}_{\mu\nu}$:

$$T^\rho{}_{\mu\nu} \equiv \frac{1}{2}\left(\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}\right) = \Gamma^\rho{}_{[\mu\nu]}\,.$$

<u>Exercise</u>: Show that torsion is a proper $\begin{pmatrix}1\\2\end{pmatrix}$ tensor. Suppose that $T^\rho{}_{\mu\nu}$ vanishes in a particular coordinate frame. What is its value in another frame related to the former one by a coord. trans. $J^\mu{}_\nu$?

– Affine Geodesics

An **affine geodesic** is defined as a special or privileged curve $x^\mu = x^\mu(u)$ along which the tangent vector $T^\mu = dx^\mu/du$ is parallelly transported into itself. On this privileged curve, the tangent vector obeys the **affine geodesic equation**:

$$\frac{DT^\mu}{Du} = \lambda(u)\, T^\mu \, .$$

Other equivalent forms of the affine geodesic equation are

$$\begin{aligned} \frac{dT^\mu}{du} + \Gamma^\mu{}_{\alpha\beta} T^\alpha T^\beta &= \lambda(u)\, T^\mu \, , \\ T^\nu \nabla_\nu T^\mu &= \lambda(u)\, T^\mu \, , \\ \frac{d^2 x^\mu}{du^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{du}\frac{dx^\beta}{du} &= \lambda(u) \frac{dx^\mu}{du} \, . \end{aligned}$$

The geodesic is called to be **affinely parameterized** if $\lambda = 0$.

Exercises:

(i) Find the affine geodesics for the plane polar coordinates (r, ϕ), knowing that the only non-zero elements of the affine connection are: $\Gamma^r{}_{\phi\phi} = -r$ and $\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = r^{-1}$.

(ii) Show that the inner product $T_\mu A^\mu$ of a parallelly transported vector A^μ, for which $T^\nu \nabla_\nu A^\mu = 0$, is preserved along a geodesic affinely parameterized.

– Metric Geodesics

A metric geodesic represents paths of extremal (shortest or longest) distance between 2 points on a Riemannian manifold.

This is equivalent to extremize the action between two points:

$$S = \int_{s_A(u_A)}^{s_B(u_B)} ds = \int_{u_A}^{u_B} du \left(g_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du} \right)^{1/2} .$$

The problem reduces to extremize the Lagrangian

$$L = \frac{ds}{du} = \left(g_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du} \right)^{1/2} ,$$

using the **Euler–Lagrange equations**:

$$\frac{d}{du}\left(\frac{\partial L}{\partial (dx^\mu/du)} \right) - \frac{\partial L}{\partial x^\mu} = 0 \, .$$

For $dL/du = 0$, we find the *affinely parameterized* **metric geodesic equation**

$$\frac{d^2 x^\mu}{du^2} + \left\{ \begin{matrix} \mu \\ \alpha\,\beta \end{matrix} \right\} \frac{dx^\alpha}{du}\frac{dx^\beta}{du} = 0 \, ,$$

where u is the **affine parameter** of the geodesic and

$$\left\{ \begin{matrix} \mu \\ \alpha\,\beta \end{matrix} \right\} = \frac{1}{2} g^{\mu\nu} \left(-\partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} \right)$$

is the **metric connection**.

Equivalence between Affine and Metric Connections

If $\nabla_\alpha g_{\mu\nu} = 0$ and the curved space is torsion-free, $T^\mu{}_{\alpha\beta} = 0$, the following equivalence relation holds true:

$$\Gamma^\mu{}_{\alpha\beta} \ = \ \left\{ \begin{matrix} \mu \\ \alpha\,\beta \end{matrix} \right\} , \quad \text{with} \quad \Gamma^\mu{}_{\alpha\beta} \ = \ \Gamma^\mu{}_{\beta\alpha} \, .$$

Because of the above equivalence relation between the two connections, one frequently uses the common name: **Christoffel connection** or **Christoffel symbol**.

Classification of Geodesics

For affinely parameterized geodesics, e.g. with $u = \tau = s/c$, the following classification holds:

$$g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du} \ = \ \text{const.} \ = \begin{cases} \ \ 0 & \text{null geodesics} \\ +1 & \text{time-like geodesics} \\ -1 & \text{space-like geodesics} \end{cases}$$

Exercises:

(i) Prove the equivalence between the affine and metric connections, if $\nabla_\alpha g_{\mu\nu} = 0$ and $T^\mu{}_{\alpha\beta} = 0$.

(ii) **Locally Inertial Coordinates**: show that by making the coordinate transformation $x'^\mu = \bar{x}^\mu + \frac{1}{2}\Gamma^\mu{}_{\alpha\beta}\,\bar{x}^\alpha \bar{x}^\beta$, where $\bar{x}^\mu = x^\mu - x^\mu_*$, one can set the transformed Christoffel connection to zero, i.e. $\Gamma'^\mu{}_{\alpha\beta} = 0$, at a given point $x^\mu = x^\mu_*$.

– Isometries and Killing's Equation

Isometry is a special coordinate transformation $x \to x'$ that leaves the metric $g_{\mu\nu}$ form-invariant, i.e. $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$.

The invariance of the line element ds^2 under an isometry implies:

$$ds^2(x') \ = \ ds^2(x) \ \Rightarrow \ g_{\mu\nu}(x) \ = \ \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu}\, g_{\alpha\beta}(x') \, .$$

If $x'^\mu = x^\mu + \varepsilon \xi^\mu(x)$ is an infinitesimal isometry, then $\xi^\mu(x)$ obeys **Killing's equation**:

$$\nabla_\mu\, \xi_\nu \ + \ \nabla_\nu \xi_\mu \ = \ 0 \, ,$$

provided that $\nabla_\rho\, g_{\mu\nu} \ = \ 0$.

The vector $\xi^\mu(x)$ is called the **Killing vector** and is associated with the symmetry of the manifold. It can be used, along with the tangent vector T_μ of a geodesic, to form the conserved quantity: $T_\mu\, \xi^\mu$.

Exercises:

(i) Prove the Killing equation stated above.

(ii) Show that if ξ^μ is an isometry, the quantity $T^\mu\, \xi_\mu$ is constant along an affinely parameterized geodesic:

$$T^\nu\, \nabla_\nu (T^\mu \xi_\mu) \ = \ 0 \, .$$

– Methods of Computing Christoffel Symbols

Given the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, there are 2 methods of evaluating the Christoffel symbols $\Gamma^\mu{}_{\alpha\beta}$:

A. Direct Computation (the hard way). Knowing the coordinate dependence of the metric $g_{\mu\nu} = g_{\mu\nu}(x^\rho)$, one may use the formula derived on p. 24:

$$\Gamma^\mu{}_{\alpha\beta} \;=\; \frac{1}{2}\, g^{\mu\nu} \left(-\partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu}\right) .$$

In 4 dimensions, one has to evaluate 40 components of the Christoffel symbol. (*Question:* How many components needs one to evaluate in n dimensions?)

B. Extremization Method (the smart way). Since geodesics are obtained by extremizing the action: $S = \int ds = \int L du$, where $L = ds/du = \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{1/2}$, one may use the Euler–Lagrange equations for L or even for $L_{\rm eff} = L^2$ (*Why?*):

$$\frac{d}{du}\left(\frac{\partial L_{\rm eff}}{\partial(dx^\mu/du)}\right) \;-\; \frac{\partial L_{\rm eff}}{\partial x^\mu} \;=\; 0$$

and compare the resulting equations with the affinely parameterized geodesic equations:

$$\frac{d^2x^\mu}{du^2} \;+\; \Gamma^\mu{}_{\alpha\beta}\,\frac{dx^\alpha}{du}\,\frac{dx^\beta}{du} \;=\; 0 \, .$$

Simple 2-dimensional example: Compute the Christoffel symbols, using method B of the line element:

$$ds^2 = \frac{1}{t^2}\left(dt^2 \;-\; dx^2\right)$$

Solution: First find $L_{\rm eff} = L^2 = ds^2/du^2$:

$$L_{\rm eff} \;=\; L^2 \;=\; \frac{\dot t^2 - \dot x^2}{t^2} \, ,$$

where $\dot t = dt/du$, $\dot x = dx/du$ and u is an affine parameter.

Then, derive the Euler–Lagrange equations for t and x. Start with calculating

$$\frac{\partial L_{\rm eff}}{\partial \dot t} = \frac{2\dot t}{t^2}\, , \qquad \frac{\partial L_{\rm eff}}{\partial \dot x} = -\frac{2\dot x}{t^2}\, , \qquad \frac{\partial L_{\rm eff}}{\partial t} = -\,\frac{2(\dot t^2 - \dot x^2)}{t^3}\, .$$

From the Euler–Lagrange equations for t and x, we get

$$\ddot t \;-\; \frac{\dot t^2}{t} \;-\; \frac{\dot x^2}{t} \;=\; 0\, , \qquad \ddot x \;-\; \frac{2}{t}\,\dot x\,\dot t \;=\; 0\, .$$

We may then read off: $\Gamma^t{}_{tt} = -1/t$, $\Gamma^t{}_{xx} = -1/t$, $\Gamma^x{}_{xt} = \Gamma^x{}_{tx} = -1/t$, whilst all other components vanish.

*Exercise**: Use method A to compute the Christoffel symbols.

Direct calculation of geodesics: Calulate the geodesics in the previous 2-dimensional example.

One may use the equivalent forms of the Lagrangian:

$$L \;=\; \frac{1}{t}\left[1 \;-\; \left(\frac{dx}{dt}\right)^2\right]^{1/2} \quad \text{or} \quad \frac{1}{t}\left[\left(\frac{dt}{dx}\right)^2 \;-\; 1\right]^{1/2} .$$

Let us use the first form for L. Since L does not depend on x, there is a first integral associated to this, given by

$$\frac{\partial L}{\partial x'} \;=\; A \;=\; \text{const.},$$

with $x' = dx/dt$. Hence, we find that

$$\frac{\partial L}{\partial x'} \;=\; \frac{x'}{t(1-x'^2)^{1/2}} \;=\; A \;\Rightarrow\; \frac{dx}{dt} \;=\; \pm\frac{At}{(1+A^2t^2)^{1/2}} .$$

The solution of this differential equation is given by

$$(Ax \;+\; B)^2 \;=\; 1 \;+\; A^2t^2 ,$$

where A, B are constants. The geodesics are hyperbolae (*Why?*).

Exercise: Find one Killing vector for the above 2-dimensional example.

4. Curvature

– Riemann Tensor

Covariant derivatives acting on a scalar commute (*Why?*):

$$[\nabla_\mu , \nabla_\nu]\, \phi \;=\; (\nabla_\mu \nabla_\nu \;-\; \nabla_\nu \nabla_\mu)\, \phi \;=\; 0 .$$

Instead, covariant derivatives acting on a vector A^ρ do *not* commute. Their commutator is given by the **Ricci identity**:

$$[\nabla_\mu , \nabla_\nu]\, A^\rho \;=\; R^\rho{}_{\alpha\mu\nu}\, A^\alpha , \quad [\nabla_\mu , \nabla_\nu]\, A_\rho \;=\; R^\alpha{}_{\rho\nu\mu}\, A_\alpha ,$$

where $R^\rho{}_{\alpha\mu\nu}$ is the **Riemann tensor**:

$$R^\rho{}_{\alpha\mu\nu} \;=\; \partial_\mu \Gamma^\rho{}_{\alpha\nu} \;-\; \partial_\nu \Gamma^\rho{}_{\alpha\mu} \;+\; \Gamma^\rho{}_{\mu\beta}\,\Gamma^\beta{}_{\alpha\nu} \;-\; \Gamma^\rho{}_{\nu\beta}\,\Gamma^\beta{}_{\alpha\mu} .$$

A **locally inertial frame (LIF)** at a given fixed point x_*^μ of the space is defined by the conditions:

$$g_{\mu\nu}(x_*) \;=\; \eta_{\mu\nu} , \qquad \left.\frac{\partial g_{\mu\nu}(x)}{\partial x^\rho}\right|_{x=x_*} \;=\; 0 ,$$

where $\eta_{\mu\nu}$ is the metric of the flat Minkowski space.

Exercise: Show by an explicit calculation that the Ricci identity stated above holds true.

Symmetries of the Riemann tensor

The Riemann tensor in a LIF at a given point x_*^μ takes on the simple form:

$$R_{\alpha\beta\mu\nu} \;=\; g_{\alpha\rho}\, R^\rho{}_{\beta\mu\nu}$$
$$= \frac{1}{2}\left(\partial_\mu\partial_\beta\, g_{\alpha\nu} + \partial_\nu\partial_\alpha\, g_{\beta\mu} - \partial_\mu\partial_\alpha\, g_{\beta\nu} - \partial_\nu\partial_\beta\, g_{\alpha\mu}\right)\Big|_{x=x_*} .$$

By inspection, we find the symmetry relations

$$R_{\alpha\beta\mu\nu} \;=\; -R_{\beta\alpha\mu\nu} \;=\; -R_{\alpha\beta\nu\mu} \;=\; R_{\mu\nu\alpha\beta}$$

and

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} \;=\; 0 .$$

Since the above symmetry relations have a tensorial covariant form, they are true in all frames, i.e. not necessarily only in LIF's.

The above notion of tensorial covariance of equations or relations is also known as **general covariance** in General Relativity.

Theorem: $R_{\alpha\beta\mu\nu} = 0 \iff$ the space is flat.

Exercise***:* Show that in n-dimensions the above symmetries of the Riemann tensor reduce the number of its components from n^4 to

$$\frac{1}{12}\, n^2\,(n^2 - 1) .$$

– Ricci Tensor and Ricci Scalar

The **Ricci tensor** is defined in terms of the Riemann tensor as

$$R_{\mu\nu} \;=\; R^\rho{}_{\mu\rho\nu} \;=\; g^{\alpha\beta}\, R_{\alpha\mu\beta\nu} .$$

Likewise, the **Ricci scalar** may be defined in terms of the Ricci tensor as follows:

$$R \;=\; g^{\mu\nu}\, R_{\mu\nu} .$$

Both the Ricci tensor and Ricci scalar are essential building blocks of the Einstein equation (see next section).

Exercises:

(i) Prove the symmetry relation for the Ricci tensor:

$$R_{\mu\nu} \;=\; R_{\nu\mu} .$$

(ii) The line element of the unit 2-sphere is given by

$$ds^2 \;=\; d\theta^2 + \sin^2\theta\, d\phi^2 .$$

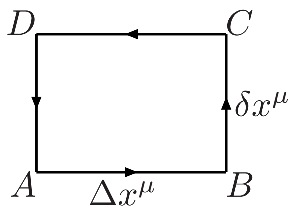
Given that the only non-zero independent component of the Riemann tensor is $R^\theta{}_{\phi\theta\phi} = \sin^2\theta$, show that the components of the Ricci tensor are given by

$$R_{\theta\theta} \;=\; 1 , \quad R_{\phi\phi} \;=\; \sin^2\theta , \quad R_{\theta\phi} \;=\; 0$$

and that the Ricci scalar R is equal to 2.

– Riemann Tensor from Parallel Transport

Consider the round trip $A \to B \to C \to D \to A$:



Making a Taylor series expansion up to second order in δx and Δx, we have

$$\mathbf{A}_B = (1 + \Delta x^\mu \nabla_\mu + \frac{1}{2} \Delta x^\mu \Delta x^\nu \nabla_\mu \nabla_\nu)\, \mathbf{A}_{A,0}\,,$$

$$\mathbf{A}_C = (1 + \delta x^\mu \nabla_\mu + \frac{1}{2} \delta x^\mu \delta x^\nu \nabla_\mu \nabla_\nu)\, \mathbf{A}_B\,,$$

$$\mathbf{A}_D = (1 - \Delta x^\mu \nabla_\mu + \frac{1}{2} \Delta x^\mu \Delta x^\nu \nabla_\mu \nabla_\nu)\, \mathbf{A}_C\,,$$

$$\mathbf{A}_{A,1} = (1 - \delta x^\mu \nabla_\mu + \frac{1}{2} \delta x^\mu \delta x^\nu \nabla_\mu \nabla_\nu)\, \mathbf{A}_D\,.$$

Upon completion of the round trip, we find

$$A^\rho_{A,1} = (1 + \delta x^\mu \Delta x^\nu [\nabla_\mu\,,\, \nabla_\nu])\, A^\rho_{A,0}$$

$$\Rightarrow \quad \Delta A^\rho = \frac{1}{2} R^\rho{}_{\alpha\mu\nu}\, A^\alpha \Delta S^{\mu\nu}\,,$$

where $\Delta S^{\mu\nu} = \delta x^\mu \Delta x^\nu - \delta x^\nu \Delta x^\mu$ represents the small area enclosed by the path.

– Bianchi Identities

Making a similar round trip along the edges of a cube this time (e.g. see textbook by L.H. Ryder, p. 120), one obtains the so-called **Bianchi identity**:

$$\sum_{\substack{\rho,\alpha,\beta \\ \text{cyclic}}} \nabla_\rho R_{\alpha\beta\mu\nu} = \nabla_\rho R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\rho\mu\nu} + \nabla_\beta R_{\rho\alpha\mu\nu} = 0$$

With the help of this, one can show the **contracted Bianchi identity**:

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0\,.$$

Exercises:

(i)* Show the validity of the Bianchi identity in a LIF.

(ii) Use the Bianchi identity to show its contracted version.

– Geodesic Deviation

Consider two infinitesimally adjacent geodesics affinely parameterized (e.g. by the proper time):

$$x^\mu \;=\; x^\mu(\tau)\,, \qquad y^\mu \;=\; x^\mu(\tau) \,+\, \delta x^\mu(\tau)\,,$$

where $\delta x^\mu(\tau)$ is the **deviation vector** connecting the two adjacent geodesics at the same proper time τ.

It can be shown that in a curved spacetime, the acceleration of the deviation vector with respect to observer's proper time τ, given by $D^2\delta x^\mu/d\tau^2$, is related to the Riemann tensor by the equation of the **geodesic deviation**:

$$\frac{D^2\,\delta x^\mu}{D\tau^2} \;=\; R^\mu{}_{\alpha\beta\rho}\, T^\alpha\, T^\beta\, \delta x^\rho\,,$$

where $T^\alpha = dx^\alpha/d\tau$ is the tangent vector of the geodesic.

Exercises:

(i) Show that the relative distance δx^i between 2 free-falling particles within a weak gravity field Φ is given by the Newtonian geodesic deviation equation

$$\frac{d^2\delta x^i}{dt^2} \;=\; (\partial^i\partial_j\,\Phi)\;\delta x^j\,.$$

(ii)** Prove the geodesic deviation equation.
[Hint: A proof of this equation may be found in the textbook by D. McMahon, p. 131–136.]

5. Einstein's Equation

– Energy-Momentum Tensor

The **energy-momentum tensor** $T^{\mu\nu}$ includes all possible forms of energy that can curve spacetime.

Specifically, if p^μ is the total 4-momentum of an energy or matter distribution, the element $T^{\mu\nu}$ represents the flux of the μ-component of the 4-momentum going through a hypersurface of spacetime for which the ν-component of x^ν is kept fixed.

Example: $T^{00} = T^{tt}$ represents the energy flow p^0 crossing an hypersurface of constant time ($x^0 = t$). An hypersurface of constant time is the volume. Hence, T^{00} is the energy density. Likewise, the time-space components T^{ti} represent the energy flux (*Why?*).

$T^{\mu\nu}$ has the following structure:

$$T^{\mu\nu} \;=\; \left(\begin{array}{l|l} T^{tt}\,:\ \text{energy density} & T^{ti}\,:\ \text{energy flux} \\ \hline T^{it}\,:\ \text{momentum density} & T^{ij}\,:\ \text{stress tensor} \end{array}\right).$$

Exercise: Show that $T^{ti} = T^{it}$ and $T^{ij} = T^{ji}$, and hence $T^{\mu\nu} = T^{\nu\mu}$.

Conservation Equations

These are derived by the relation

$$\nabla_\nu T^{\mu\nu} \;=\; 0 \,.$$

In a LIF, this simplifies to the SR result:

$$\partial_\nu T^{\mu\nu} \;=\; 0 \,.$$

For $\mu = 0$, one finds the **continuity equation** for **energy conservation**:

$$\frac{\partial T^{tt}}{\partial t} + \frac{\partial T^{it}}{\partial x^i} \;=\; \frac{\partial \varepsilon}{\partial t} + \nabla \cdot \boldsymbol{\pi} \;=\; 0 \,,$$

where $\varepsilon = T^{tt}$ and $\pi^i = T^{it}$ are the energy and momentum densities, respectively.

Exercise: Show that for the spatial components $\mu = i$, one obtains the analog of Newton's 2nd law:

$$\frac{\partial \boldsymbol{\pi}}{\partial t} \;=\; \boldsymbol{\phi} \,,$$

where $\phi^i \;=\; -\partial T^{ij}/\partial x^j$ is the exerted force on an energy-matter distribution per unit volume.

Perfect fluid is a fluid with no viscosity and heat conduction. Its energy-momentum tensor in SR or in a LIF is given by

$$T^{\mu\nu} \;=\; (\rho + P)\, u^\mu u^\nu \;-\; P\, \eta^{\mu\nu} \,,$$

where ρ and P are the energy density and the pressure of the fluid, respectively, and $u^\mu = \gamma(1\,,\ \mathbf{v})$ is the 4-velocity.

Generalization to the curved spacetime:

$$T^{\mu\nu} \;=\; (\rho + P)\, u^\mu u^\nu \;-\; P\, g^{\mu\nu} \,.$$

Exercises:

(i) Show that for a perfect fluid at rest in a LIF, its energy-momentum tensor reads: $T^{\mu\nu} \;=\; \mathrm{diag}\,(\rho, P, P, P)$.

(ii) Use the conservation equation for the energy-momentum tensor to show that for a non-relativistic pressureless perfect fluid ($P = 0$) in a LIF, the continuity equation for the energy conservation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \;=\; 0 \,.$$

(iii)* Given that $T_{\mu\nu} \;=\; F_{\mu\rho} F^{\rho}{}_{\nu} + \frac{1}{4}\, g_{\mu\nu}\, F^{\alpha\beta} F_{\alpha\beta}$ is the energy-momentum tensor of the electromagnetic field, calculate the components T^{00} and T^{0i} in a LIF and discuss their physical meaning.

– Einstein's Equation

Intuitive approach to deriving Einstein's equation:

	Newtonian Gravity	General Relativity
What mass does	Produces a field Φ causing a force on other mass m $\mathbf{F} \;=\; -m\,\nabla\Phi$	Curves spacetime $ds^2 \;=\; g_{\mu\nu}(x)\,dx^\mu dx^\nu$
Motion of a particle	Newton's 2nd law $\frac{d^2x^i}{dt^2} \;=\; -\delta^{ij}\,\frac{\partial\Phi}{\partial x^j}$	Geodesic equation $\frac{d^2x^\mu}{d\tau^2} \;=\; -\,\Gamma^\mu{}_{\nu\lambda}\,\frac{dx^\nu}{d\tau}\,\frac{dx^\lambda}{d\tau}$
Field equation	Poisson's equation $\nabla^2\Phi \;=\; 4\pi G\rho_m$	Einstein's equation $G_{\mu\nu} \;=\; 8\pi G T_{\mu\nu}$

Einstein's equation is given by

$$G_{\mu\nu} \;\equiv\; R_{\mu\nu} \;-\; \frac{1}{2}\,g_{\mu\nu}\,R \;=\; 8\pi G\,T_{\mu\nu}\;.$$

Both the left and right side of this equation satisfy the conservation conditions (*Why?*):

$$\nabla^\mu\,G_{\mu\nu} \;=\; 0\;, \qquad \nabla^\mu\,T_{\mu\nu} \;=\; 0\;.$$

Exercise: What else can be added to Einstein's equation in agreement with the conservation conditions stated above?

– Newtonian Limit

The geodesic of a non-relativistic (NR) particle with $dx^i/dx^0 \ll 1$ (or $dx^i/d\tau \ll dx^0/d\tau$) is

$$\frac{d^2x^\mu}{d\tau^2} \;+\; \Gamma^\mu{}_{00}\left(\frac{dx^0}{d\tau}\right)^2 \;+\; \mathcal{O}(dx^i/d\tau) \;=\; 0\;.$$

Consider the weak gravitational field approximation for $g_{\mu\nu}$:

$$g_{\mu\nu}(x) \;=\; \eta_{\mu\nu} \;+\; h_{\mu\nu}(x)\;,$$

where $h_{\mu\nu}(x) \ll 1$ and independent of time, i.e. $\partial_0 h_{\mu\nu} = 0$.

To leading order in $h_{\mu\nu}$, $\Gamma^\mu{}_{00}$ is given by (see p. 24)

$$\Gamma^\mu{}_{00} \;=\; -\,\frac{1}{2}\,\eta^{\mu\nu}\,\partial_\nu h_{00}\;.$$

Since $\Gamma^0{}_{00} = 0$, the geodesic equation implies for the time component $\mu = 0$ that $d^2t/d\tau^2 = 0 \;\Rightarrow\; dt/d\tau =$ const. Eliminating τ in favour of t in the geodesics for the spatial components, one obtains

$$\frac{d^2\mathbf{x}}{dt^2} \;=\; -\,\frac{1}{2}\,(\nabla h_{00})\;.$$

Comparing this with Newton's 2nd law, we find $h_{00} = 2\Phi$ and

$$g_{00} \;=\; 1 \;+\; 2\Phi\;.$$

Alternative form of Einstein's equation:

$$R_{\mu\nu} \;=\; 8\pi G \left(T_{\mu\nu} \;-\; \frac{1}{2}\, g_{\mu\nu} T \right) \;,$$

where $T = g^{\mu\nu} T_{\mu\nu}$.

The above alternative form is useful to obtain Poisson's equation in the Newtonian limit of Einstein's equation. Given that $p \ll \rho$ in the NR limit, the energy-momentum tensor simplifies to $T_{\mu\nu} \;=\; \mathrm{diag}\,(\rho_m\,,\; 0\,,\; 0\,,\; 0)$, where ρ_m is the rest mass density.

Exercises:

(i) Contract the Einstein equation with $g^{\mu\nu}$ to obtain the relation:

$$R \;=\; -8\pi G\, T \;.$$

Substitute R into Einstein's equation to find its alternative form.

(ii) Calculate the component of the Ricci tensor R_{00} in the approximation of a stationary weak gravitational field to find that

$$R_{00} \;=\; \frac{1}{2}\nabla^2 h_{00} \;+\; \mathcal{O}(h^2_{\mu\nu}) \;.$$

Use then the alternative form of Einstein's equation, together with the fact that $h_{00} = 2\Phi$, to derive Poisson's equation in the Newtownian limit: $\nabla^2\Phi \;=\; 4\pi G \rho_m$.

– Gravitational Radiation

Linearized gravity:

$$g_{\mu\nu}(x) \;=\; \eta_{\mu\nu} \;+\; h_{\mu\nu}(x)\,, \qquad g^{\mu\nu}(x) \;=\; \eta^{\mu\nu} \;-\; h^{\mu\nu}(x)\;.$$

Weak field approximation: $h_{\mu\nu} \ll 1$, $g^{\mu\nu}\, g_{\nu\lambda} = \delta^{\mu}{}_{\lambda} + \mathcal{O}(h^2)$.

Coord. trans. $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, with $\epsilon^\mu \ll 1$:

$$J^{\mu}{}_{\nu} \;=\; \delta^{\mu}{}_{\nu} \;+\; \partial_\nu \epsilon^\mu\,, \quad (J^{-1})^{\mu}{}_{\nu} \;=\; \delta^{\mu}{}_{\nu} \;-\; \partial_\nu\epsilon^\mu \;+\; \mathcal{O}(\epsilon^2)\;.$$

Metric changes under coord. trans.:

$$\eta_{\mu\nu} \;+\; h'_{\mu\nu} \;=\; \eta_{\mu\nu} \;+\; (h_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu) \;+\; \mathcal{O}\,(\epsilon^2)$$

General covariance: Einstein's equation does not change its form under coord. trans.: $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu$
Coord. trans. are also called **gauge transformations**.

Lorentz gauge: $\partial_\mu \bar{h}^{\mu\nu} = 0$, where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ and $h = \eta^{\alpha\beta} h_{\alpha\beta}$.

Gravity wave equations from linearized gravity*:

$$\partial^2 h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T \right)\,, \quad \partial^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\;.$$

Exercise: Given $h_{\mu\nu}$, find ϵ_μ, such that $h'_{\mu\nu}$ obeys the condition of the Lorentz gauge.

Linearized Gravity and Electromagnetism

	Linearized Gravity	Electromagnetism
Field equation	Einstein's equation with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$	Maxwell's equations
Basic potentials	Linearized metric $h_{\mu\nu}(x)$	4-vector potential $A^\mu = (\Phi\,,\ \mathbf{A})$
Sources	Energy-momentum tensor $T^{\mu\nu}$	4-vector current $J^\mu = (\rho\,,\ \mathbf{J})$
Lorentz gauge	$\partial_\mu \bar{h}^{\mu\nu} = 0$ $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$	$\partial_\mu A^\mu = 0$
Wave equation with source*	$\partial^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$	$\partial^2 A^\nu = \mu_0 J^\nu$
Solution*	$\bar{h}^{ij} = 4G \int d^3x' \frac{[T^{ij}]_{\rm ret}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$\mathbf{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{[\mathbf{J}]_{\rm ret}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$
Large r, long-wavelength approximation*	$\bar{h}^{ij} = \frac{2G\,[\ddot{I}^{ij}]_{\rm ret}}{r}$ $I^{ij} = \int d^3x\ \rho_m\, x^i x^j$	$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{[\dot{\mathbf{d}}]_{\rm ret}}{r}$ $\mathbf{d} = \int d^3x\ \rho\,\mathbf{x}$
Time-averaged radiated power*	$P = \frac{G}{5}\langle \dddot{I}_{ij}\dddot{I}^{ij}\rangle$	$P = \frac{\mu_0}{6\pi}\langle \ddot{\mathbf{d}}^2\rangle$

For further reading, see J.B. Hartle, Sections 16 and 23.

6. Schwarzschild Solution

– Spherically Symmetric Vacuum Solution to Einstein's Equation

Einstein's equation in vacuum ($T^{\mu\nu} = 0$) simplifies to (*Why?*):

$$R_{\mu\nu} = 0\,.$$

We seek a spherically symmetric solution to this equation through the line element:

$$ds^2 = e^{\nu(r,t)}\,dt^2 - e^{\lambda(r,t)}\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\,.$$

The non-zero components of $R_{\mu\nu}$ are (with $' \equiv \frac{\partial}{\partial r}$ and $\dot{} \equiv \frac{\partial}{\partial t}$)

$$\begin{aligned}
R_{tt} &= \frac{1}{2}e^{-\lambda}\left[\nu'' + \frac{1}{2}\nu'\left(\nu'-\lambda'\right) + \frac{2\nu'}{r}\right] \\
&\quad + e^{-\nu}\left[\dot{\lambda}\left(\dot{\nu}-\dot{\lambda}\right) - \frac{1}{2}\ddot{\lambda}\right]\,, \\
R_{tr} &= \frac{\dot{\lambda}}{2r}\,, \\
R_{rr} &= \frac{1}{2}e^{-\nu}\left[\ddot{\lambda} - \frac{1}{2}\dot{\lambda}\left(\dot{\nu}-\dot{\lambda}\right)\right] \\
&\quad - \frac{1}{2}e^{-\lambda}\left[\nu'' + \frac{1}{2}\nu'\left(\nu'-\lambda'\right) - \frac{2\lambda'}{r}\right]\,, \\
R_{\theta\theta} &= 1 - e^{-\lambda}\left[1 + \frac{1}{2}r\left(\nu'-\lambda'\right)\right]\,, \\
R_{\phi\phi} &= \sin^2\theta\,R_{\theta\theta}\,.
\end{aligned}$$

$$R_{tr} = 0 \Rightarrow \lambda = \lambda(r) ,$$
$$R_{tt} + R_{rr} = 0 \Rightarrow \nu' + \lambda' = 0 \Rightarrow \nu + \lambda = f(t) .$$

Since $f(t)$ can be set to zero by a time coord. trans.: $dt' = e^{f(t)/2}dt$, we find that

$$\nu(r) = -\lambda(r) .$$

$$R_{\theta\theta} = 0 \Rightarrow (r\, e^{\nu})' = 1 \Rightarrow e^{\nu} = 1 + \frac{C}{r} .$$

In the Newtonian limit,

$$g_{00} \approx 1 + 2\Phi = 1 - \frac{2GM}{r} \Rightarrow C = -2GM .$$

Hence, the vacuum solution to the Einstein equation exterior to an object of mass M is given by the line element of the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) .$$

Exercise: Find 2 Killing vectors related to the Schwarzschild metric.

– Birkhoff's Theorem

A spherically symmetric vacuum solution exterior to a mass distribution must be static and the metric must have the form of the Schwarzschild solution.

Properties of the Schwarzschild metric:

- It is spherically symmetric.
- It is static. The metric is invariant under reflections: $t \to -t$ and translations $t \to t +$ const.
- It is asymptotically flat. It goes over to Minkowski metric as $r \to \infty$.
- g_{tt} and g_{rr} flip sign at $r = r_s = 2GM$, which is called the **Schwardschild radius** or the **event horizon**.

Exercises:

(i) Which conditions on the Ricci tensor imply that a spherically symmetric vacuum solution outside of a mass distribution has to be static?

(ii) Can a pulsating spherical star emit gravitational waves and why?

– Gravitational Redshift

Consider different radial slices $r = \text{const}$ of the Schwarzschild metric: $ds^2 = g_{tt}(r)\, dt^2$.

The frequency of light ν is inversely proportional to the proper clock time at this location:

$$\nu \;\propto\; \frac{1}{\Delta\tau(r)} \;=\; \frac{c}{\Delta s(r)} \;,$$

where $\Delta\tau(r) = \sqrt{g_{tt}(r)}\, \Delta t$ and $\Delta t = \text{const}$.

At two different radial locations $r_1 = \text{const}$ and $r_2 = \text{const}$, the two light frequencies are related by

$$\frac{\nu_1}{\nu_2} \;=\; \frac{\Delta\tau(r_2)}{\Delta\tau(r_1)} \;=\; \sqrt{\frac{g_{tt}(r_2)}{g_{tt}(r_1)}} \;.$$

Exercise: Show that for a weak gravitational field $\Phi(r)$, the ratio of frequencies is approximately given by

$$\frac{\nu_1}{\nu_2} \;=\; 1 \;-\; \Phi(r_1) \;+\; \Phi(r_2) \;.$$

Compare this result with the one derived on p. 11.

– Dynamics in the Schwarzschild Spacetime

Consider the effective Lagrangian for the Schwarzschild metric

$$L_{\rm eff} \;=\; \left(1 - \frac{r_s}{r}\right) \dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 - r^2 \left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) ,$$

where $\dot{} \equiv \frac{d}{d\tau}$ and $r_s \equiv 2GM$.

There are 3 first integrals that result from $L_{\rm eff}$:

$$\frac{\partial L_{\rm eff}}{\partial t} \;=\; 0 \;\;\Rightarrow\;\; \frac{\partial L_{\rm eff}}{\partial \dot{t}} \;=\; 2\left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} \;=\; 2\,\varepsilon \;,$$

$$\frac{\partial L_{\rm eff}}{\partial \phi} \;=\; 0 \;\;\Rightarrow\;\; \frac{\partial L_{\rm eff}}{\partial \dot{\phi}} \;=\; 2\, r^2 \sin^2\theta\, \frac{d\phi}{d\tau} \;=\; 2\,\ell \;,$$

$$L_{\rm eff} \;=\; \text{const.} \;=\; K \;=\; \begin{cases} \;\;0 & \text{null geodesics} \\ +1 & \text{time-like geodesics} \\ -1 & \text{space-like geodesics} \end{cases} \;,$$

where ε and ℓ are constants of integration related to the conserved energy and angular momentum per unit rest mass, respectively.

For $\theta = \text{const} = \pi/2$, all first integrals are related through

$$K \;=\; \left(1 - \frac{r_s}{r}\right)^{-1} \left[\varepsilon^2 \;-\; \left(\frac{dr}{d\tau}\right)^2\right] \;-\; \frac{\ell^2}{r^2} \;.$$

Exercise: Find the geodesics and Christoffel symbols for the Schwarzschild spacetime.

– Particle and Photon Orbits

After rearranging the last eqn. on p. 48, we find (for $r \geq r_s$)

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \underbrace{\frac{\varepsilon^2 - K}{2}}_{=E} - \underbrace{\left[\frac{\ell^2}{2r^2}\left(1 - \frac{r_s}{r}\right) - \frac{K r_s}{2r}\right]}_{=V_{\text{eff}}} .$$

I. Particle orbits $K = 1$:

$$V_{\text{eff}} = \frac{\ell^2}{2r^2}\left(1 - \frac{r_s}{r}\right) - \frac{r_s}{2r} = \frac{\ell^2}{2r^2}\left(1 - \frac{2GM}{r}\right) - \frac{GM}{r}$$

(a) $\boldsymbol{\ell < \sqrt{3} r_s}$. No stable circular orbit exists. Particle crushes into origin if $E < 0$. Particle escapes (from $r = r_s$) if $E > 0$.

(b) $\boldsymbol{\sqrt{3} r_s < \ell < 2 r_s}$. $V_{\text{eff}}(r)$ has a maximum $r_{\max}$ and a minimum $r_{\min}$, with $r_{\max} < r_{\min}$. There is a *stable* circular orbit at $r_{\min}$ and an *unstable* circular one at $r_{\max}$. Particle crushes into the origin if $E < 0$, while it escapes if $E > 0$.

(c) $\boldsymbol{\ell > 2 r_s}$. If $E > V_{\max}$, particle escapes from $r = r_s$. For $E < V_{\max}$, particle coming from $r = \infty$ gets repelled back. Most astonishingly, a particle from $r = \infty$, with $E > V_{\max}$, will plunge into the center.

II. Photon orbits $K = 0$:

$$V_{\text{eff}} = \frac{\ell^2}{2r^2}\left(1 - \frac{r_s}{r}\right) = \frac{\ell^2}{2r^2}\left(1 - \frac{2GM}{r}\right) .$$

V_{eff} has the maximum value

$$V_{\max} = \frac{2\,\ell^2}{27\, r_s^2} ,$$

at $r = \frac{3}{2} r_s$.

If $E < V_{\max}$, a photon from $r = \infty$ gets repelled. In this case, a photon running out from $r = r_s$ will fall back to the center. If $E > V_{\max}$, a photon coming from $r = \infty$ will plunge into the origin [compare this with I(c)]. Finally, for $E = V_{\max}$, the photon has a circular *unstable* orbit.

Exercises:

(i) Find $r_{\min}$ and $r_{\max}$, as well as $V_{\min}$ and $V_{\max}$, for the cases I(b) and I(c) of particle orbits.

(ii)** Calculate the radial plunge orbit $r = r(t)$ for the case I(c).

[Hint. The result may be found in the textbook by J.B. Hartle, on p. 199.]

– Light Deflection

The shape of bound orbits: Dividing $d\phi/d\tau = \ell/r^2$ given on p. 48 by $dr/d\tau$ on p. 49, one finds the familiar result from Classical Mechanics:

$$\frac{d\phi}{dr} = \pm \frac{1}{r^2} \frac{\ell}{\sqrt{2}\,[E - V_{\rm eff}(r)]^{1/2}} .$$

Note that this equation holds for light and test masses alike.

For light rays coming in from infinity with an impact parameter $d = \ell/\sqrt{2E}$ and going out to infinity, the total deflection angle is twice the angle swept out from the turning point $r = r_{\rm min}$ to $r = \infty$:

$$\Delta\phi = 2 \int_{r_{\rm min}}^{\infty} \frac{dr}{r^2} \left[\frac{1}{d^2} - \frac{1}{r^2} \left(1 - \frac{r_s}{r}\right) \right]^{-1/2} .$$

Upon introducing the dimensionless parameter $w = d/r$, the $\Delta\phi$ may be cast into the form (*Why?*):

$$\Delta\phi = 2 \int\limits_0^{w_{\rm max}} dw \left(1 - \frac{r_s}{d}\, w\right)^{-1/2} \left[\left(1 - \frac{r_s}{d}\, w\right)^{-1} - w^2 \right]^{-1/2} .$$

Exercise: Show that the impact parameter of light rays coming from infinity is given by $d = \ell/\sqrt{2E}$.

To leading order in $r_s/d \ll 1$, $\Delta\phi$ is given by (*Why?*)

$$\Delta\phi = 2 \int\limits_0^{w_{\rm max}} dw \, \frac{1 + \left(\frac{r_s}{2d}\right) w}{\left[1 + \left(\frac{r_s}{d}\right) w - w^2\right]^{1/2}} \approx \pi + 2\,\frac{r_s}{d}$$

Hence, the predicted deflection angle $\delta\phi_{\rm defl}$ of light rays is

$$\delta\phi_{\rm defl} = \Delta\phi - \pi \approx \frac{2\,r_s}{d} \quad (\text{for } r_s/d \ll 1) ,$$

where d is the impact parameter and $r_s = 2GM$ is the Schwarzschild radius of a spherical object of mass M.

Exercises:

(i) Compute the maximal value of r_s/d for our Sun. What is the minimum value of d?

(ii)* Use look-up tables or otherwise to verify the result for the above integral for $\Delta\phi$ to leading order in r_s/d.

(iii) Estimate that the deflection angle of distant light rays grazing the limb of the Sun is $\delta\phi_\odot \approx 1.75''$, where $1'' = (\pi/648\,000)$ rads.

– Perihelion Precession

The angle of a complete revolution is given by

$$\Delta\phi \;=\; 2\ell \int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2} \left[2E \;-\; \frac{\ell^2}{r^2}\left(1 - \frac{r_s}{r}\right) \;+\; \frac{r_s}{r} \right]^{-1/2} ,$$

where $r_{\min}$ and $r_{\max}$ are the two turning points of the elliptic orbit.

Using the fact that $E = (\varepsilon^2 - 1)/2$ and changing the integration variable to $u = 1/r$, we may rewrite $\Delta\phi$ as

$$\begin{aligned} \Delta\phi \;&=\; 2 \int_{u_{\min}}^{u_{\max}} du\, (1 - r_s u)^{-1/2} \\ &\times \left[\frac{\varepsilon^2}{\ell^2} (1 - r_s u)^{-1} - \frac{1}{\ell^2} \;-\; u^2 \right]^{-1/2} . \end{aligned}$$

The above integral may be approximately computed in powers of r_s/ℓ to give

$$\Delta\phi \;=\; 2\pi \;+\; \frac{3\pi\, r_s^2}{2\,\ell^2} \,.$$

Hence, the precession of the perihelion is given by

$$\delta\phi_{\rm prec} \;=\; \Delta\phi \;-\; 2\pi \;=\; \frac{3\pi\, r_s^2}{2\,\ell^2} \;=\; \frac{6\pi\, G^2 M^2}{\ell^2} \,.$$

*Exercise**: Steps for the computation of $\delta\phi_{\rm prec}$:

(i) Expand in powers of r_s/ℓ the integrand in the integral for $\Delta\phi$ to obtain

$$\begin{aligned} \Delta\phi \;=\; &\left(1 + \frac{\varepsilon^2 r_s^2}{2\ell^2}\right) 2 \int_{u_{\min}}^{u_{\max}} du \left[\frac{\varepsilon^2}{\ell^2} (1 + r_s u) - \frac{1}{\ell^2} - u^2 \right]^{-1/2} \\ &+ r_s \int_{u_{\min}}^{u_{\max}} du\; u \left[\frac{\varepsilon^2}{\ell^2} (1 + r_s u) - \frac{1}{\ell^2} - u^2 \right]^{-1/2} , \end{aligned}$$

where $u_{\min,\max}$ are the roots of the quadratic equation in u contained in $[\cdots]^{-1/2}$.

(ii) Use look-up tables to calculate the 2 integrals. In particular, you should find that the first integral is equal to π, while the second one is $(\pi/2)\,(u_{\min} + u_{\max}) = \pi\varepsilon^2 r_s/(2\ell^2)$.

(iii) Put all these intermediate results together and use the fact that $\varepsilon^2 \approx 1$ for a NR planetary motion to deduce $\Delta\phi$ given on the previous page, and so $\delta\phi_{\rm prec}$.

– Black Holes

Consider the t-r part of the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 ,$$

where $d\theta = d\phi = 0$. The metric has a singularity at $r = r_s = 2GM$, i.e. $g_{rr} \to \infty$ as $r \to r_s$. However, invariants under coordinate transformations, such as R, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta}$, do not display such a singularity. The invariant

$$R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} = \frac{6r_s^2}{r^6}$$

is regular at $r = r_s$. Such a singularity depends on the choice of the coordinate system and is called **axis singularity**. It can be removed by changing to another coordinate system.

The Schwarzschild metric contains a true (non-removable) singularity as $r \to 0$ (*Why?*)

To remove the axis singularity, we introduce the so-called **tortoise coordinate**

$$r_* = r + r_s \ln\left(\frac{r}{r_s} - 1\right) .$$

Exercise: Show that null geodesics are described by the curves: $t = \pm r_* + \text{const.}$ What is the physical meaning of the $\pm$ sign?

Eddington–Finkelstein coordinates:

$$u = t - r_* , \qquad v = t + r_* .$$

In terms of u and v, the Schwarzschild metric takes on the simpler form

$$ds^2 = \left(1 - \frac{r_s}{r}\right) du\, dv - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) .$$

Evidently, the metric is not singular at $r = r_s$.

The curves $u = \text{const.}$ represent outgoing null geodesics, while $v = \text{const.}$ represent ingoing null geodesics (*Why?*).

At $r = r_s$, there is an one-way membrane called the **event horizon**, where future-directed time-like and light-like curves (or particles) can cross from $r > r_s$ to $r < r_s$, but the reverse is not possible (*Why?*).

Light cones tip over by 90° when going from $r > r_s$ to $r < r_s$. For $r < r_s$, future light cones point to the origin $r = 0$ (*Why?*).

The fate of stars: A star of mass M may have one of the following end phases:

$$\begin{aligned} M \;&\lesssim\; 1.4\, M_\odot &&\Rightarrow \text{white dwarf} \\ 1.4\, M_\odot \lesssim M \;&\lesssim\; 5\, M_\odot &&\Rightarrow \text{neutron star} \\ M \;&\gtrsim\; 5\, M_\odot &&\Rightarrow \text{creation of a black hole is possible} \end{aligned}$$

7. Friedmann–Robertson–Walker Universe

– Expansion, Isotropy and Homogeneity

Astronomical observations of densities of galaxies, background radiation and vacuum energy suggest that our Universe is uniformly expanding in all directions.

The **Friedmann–Robertson–Walker (FRW)** cosmological model is based on the **cosmological principle** that on large scales our Universe is **homogeneous** and **isotropic**.

At each epoch, a homogeneous universe is the same from place to place, and an isotropic one is the same in one direction as in any other.

Schur's theorem (without proof): Globally isotropic n-dimensional spaces ($n > 2$) are manifolds of constant curvature k. The Riemann tensor for an isotropic space is given by

$$R_{\mu\nu\alpha\beta} \;=\; k\,(\,g_{\mu\alpha}\,g_{\nu\beta} \;-\; g_{\mu\beta}\,g_{\nu\alpha}\,)\;,$$

where k is constant on every point of the manifold.

Exercise: Show that the Riemann tensor stated above for an isotropic space satisfies all the symmetry relations derived on p. 31. Is a term proportional to $\varepsilon_{\mu\nu\alpha\beta}$ allowed by the symmetry relations?

– FRW Metric

The FRW metric incorporates the cosmological principle of isotropy and homogeneity. The line element of the FRW metric has the general form

$$ds^2 \;=\; dt^2 \;-\; a^2(t)\,d\sigma^2\;,$$

where $d\sigma^2$ is the line element of a 3-dim. space of constant curvature and $a(t)$ is a scale factor that implements the evolution in time of the 3-dim. space.

If $\dot a \equiv da/dt > 0$, the FRW metric describes an expanding universe. The rate of expansion of our Universe is determined by the **Hubble parameter** H, which is defined as

$$H \;=\; \frac{\dot a}{a}\;.$$

The present value of H is $H_0 = 73\,^{+4}_{-3}$ (km/s)/Mpc.

The line element of the FRW metric takes on the following explicit form:

$$ds^2 \;=\; dt^2 \;-\; a^2(t)\,\left[\frac{dr^2}{1-kr^2} \;+\; r^2\,(d\theta^2 \;+\; \sin^2\theta\,d\phi^2\,)\right]\;,$$

where $k \;=\; +1,\,0\,,\,-1$ (in arbitrary G^{-1} units) are the curvatures for a **closed**, **flat** and **open** universe, respectively.

Exercise: Find all Killing vectors associated with the FRW metric.

Other forms of the FRW metric:

Substituting $r = \sin\chi\,,\ \ \chi\,,\ \ \sinh\chi$ for $k = +1, 0\,, -1$, respectively, into the FRW metric, we find the alternative form

$$ds^2 \;=\; dt^2 - a^2(t)\left[d\chi^2 + \left\{ \begin{array}{c} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{array} \right\} (d\theta^2 + \sin^2\theta\, d\phi^2\,) \right] .$$

The spatial part of the metric describes a **closed** 3-sphere for $k = +1$, **flat** space for $k = 0$ and an **open** 3-hyperbola for $k = -1$ (*Why?*).

Introducing a new time coordinate $\boldsymbol{\eta}$, the so-called **conformal time**, given by

$$a(t)\, d\eta \;=\; dt \quad\Rightarrow\quad \eta \;=\; \int_0^t \frac{dt'}{a(t')}\ ,$$

the FRW metric can be brought into the form:

$$ds^2 \;=\; a^2(t)\left[d\eta^2 \;-\; \frac{dr^2}{1-kr^2} \;-\; r^2\,(d\theta^2 \;+\; \sin^2\theta\, d\phi^2\,)\right] .$$

The FRW metric then becomes conformal to that of the Minkowski spacetime for $k = 0$ (*Why?*).

*Exercise**:* Derive the explicit spatial form $d\sigma^2$ of the FRW metric given on the previous page, by modifying appropriately the ansatz for the Schwarzschild metric and using Schur's theorem.

– Friedmann and Raychaudhuri Equations

For the FRW spacetime, the elements of the Ricci tensor $R_{\mu\nu}$ are (with $\dot{} \equiv \frac{d}{dt}$)

$$\begin{aligned} R_{00} &= -\frac{3\ddot{a}}{a}\ , \\ R_{0i} &= R_{i0} \;=\; 0\ , \\ R_{ij} &= -\left(\frac{2k}{a^2} + \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} \right) g_{ij}\ , \end{aligned}$$

where $g_{ij} = -\,a^2(t)\ \mathrm{diag}\left[(1-kr^2)^{-1},\, r^2,\, r^2\sin^2\theta\,\right]$ are the spatial components of the FRW metric (*Question*: What is g_{00}?).

Consider the alternative form of Einstein's eqn (see p. 41),

$$R_{\mu\nu} \;=\; 8\pi G \left(T_{\mu\nu} \;-\; \frac{1}{2}\, g_{\mu\nu} T \right)\ ,$$

where $T = T^\lambda_{\ \lambda}$.

According to **Weyl's postulate**, our Universe may well be described by a perfect fluid. In this case, the RHS of the above equation becomes (*Why?*)

$$T_{\mu\nu} \;-\; \frac{1}{2}\, g_{\mu\nu} T \;=\; (\rho + P)\, u_\mu u_\nu \;-\; \frac{1}{2}\, g_{\mu\nu}\,(\rho - P)\ .$$

For a locally comoving frame with $u^\mu = (1,0,0,0)$, the Einstein eqn for R_{00} gives the **Raychaudhuri equation**:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)\,.$$

Einstein's eqn for R_{ij} leads to the **Friedmann equation**:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,.$$

From the conservation equation $\nabla_\nu T^{\mu\nu} = 0$ (with $\mu = 0$), the following **energy conservation equation** can be derived:

$$\frac{\partial\rho}{\partial t} = -3H(\rho + P)\,.$$

Raychaudhuri's, Friedmann's and energy conservation eqs are not all independent of each other (*Why?*).

Exercises:

(i) Use Einstein's eqn in its alternative form to prove Raychaudhuri's and Friedmann's equations stated above.

(ii) Given that ${\Gamma^t}_{tt} = 0$, ${\Gamma^r}_{tr} = {\Gamma^\theta}_{t\theta} = {\Gamma^\phi}_{t\phi} = \dot{a}/a = H$, derive the energy conservation equation given above.

(iii) Show that the energy conservation equation is equivalent to the 1st law of thermodynamics for a perfect fluid: $d(\rho V) + P dV = 0$, where V is the 3-volume.

– Matter-, Radiation-, and Vacuum Energy- Dominated Universes

A **matter-dominated** universe is **pressureless**, i.e. $P = 0$. Hence, the energy conservation equation becomes

$$\frac{\partial\rho}{\partial t} = -3H\rho \quad\Rightarrow\quad \rho = \frac{\rho_0}{a^3}\,,$$

where ρ_0 is the initial energy density.

The equation of state for **radiation** is $\rho = 3P$, as derived from the energy-momentum tensor of electromagnetism (see exercise on p. 38). For a **radiation-dominated** universe , we obtain (*Why?*)

$$\frac{\partial\rho}{\partial t} = -4H\rho \quad\Rightarrow\quad \rho = \frac{\rho_0}{a^4}\,.$$

The equation of state for **vacuum energy** is $\rho = -P$. For a **vacuum energy-dominated** universe, the energy density remains constant, i.e. $\rho = \rho_0 =$ const (*Why?*).

The present energy density ρ_c for a flat FRW model ($k = 0$) is called the **critical density** and is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} = 10.54\,h^2\ \mathrm{keV/cm^3},$$

where $H_0 = 100\,h\ \mathrm{(km/s)/Mpc}$, with $h = 0.73\ ^{+0.04}_{-0.03}$, is today's Hubble's constant value.

The present matter, radiation and vacuum energy densities normalized to ρ_c are defined as

$$\Omega_m = \frac{\rho_m(t_0)}{\rho_c}, \quad \Omega_r = \frac{\rho_r(t_0)}{\rho_c}, \quad \Omega_v = \frac{\rho_v(t_0)}{\rho_c}.$$

This definition may be extended to a normalized curvature density given by

$$\Omega_k = -\frac{k}{H_0^2 a_0^2},$$

where the standard convention is that $a_0 = a(t_0) = 1$.

With the above definitions, the Friedmann equation evaluated at the present epoch becomes (*Why?*)

$$\Omega_m + \Omega_r + \Omega_v + \Omega_k = 1.$$

Exercises:

(i) Show that the total energy density of our Universe is given by

$$\rho(a) = \rho_c \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_v \right).$$

(ii) Derive the **alternative form of Friedmann's equation**:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_v + \frac{\Omega_k}{a^2}.$$

Time Evolution of the FRW Universe:

The time evolution of the scale factor $a(t)$ is determined by the alternative form of Friedmann's equation (*Why?*):

$$\dot{a} = H_0 \sqrt{\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_v\, a^2 + \Omega_k}.$$

The age of a FRW universe may also be determined by this equation:

$$t_0 = \int_0^{a_0=1} \frac{da}{\dot{a}} = \frac{1}{H_0} \int_0^1 \frac{da}{\sqrt{\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_v\, a^2 + \Omega_k}}.$$

Exercises:

(i) Show that the time dependences of $a(t)$ are given by

$$a(t) \propto t^{2/3}, \quad t^{1/2}, \quad \exp(H_0 \Omega_v^{1/2} t), \quad t,$$

for a matter-, radiation-, vacuum energy- and curvature-dominated (with $k = -1$) FRW universe, respectively.

(ii) Calculate the age of a flat matter-dominated universe to find that

$$t_0 = \frac{2}{3\, H_0}.$$

Closed, Flat and Open FRW Universes:

Consider a matter-dominated FRW universe with non-zero curvature (i.e. set $\Omega_r = \Omega_v = 0$). To evaluate the time evolution of such a universe, we use the conformal time

$$\eta \;=\; \int_0^t \frac{dt}{a} \;=\; \int_0^a \frac{da}{a\,\dot a} \;=\; \frac{1}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_m\, a \,+\, \Omega_k\, a^2}}\;,$$

with the constraint $\Omega_k + \Omega_m = 1$ (*Why?*).

Using look-up integral tables, or otherwise, we find

- for $\boldsymbol{\Omega_k > 0}$ ($k = -1$):

$$a(\eta) \;=\; \frac{\Omega_m}{2(1-\Omega_m)}\left[\cosh\left(\sqrt{1-\Omega_m}\,H_0\,\eta\right) \;-\; 1\right]$$

  This is the solution for **an ever expanding universe**.

- for $\boldsymbol{\Omega_k < 0}$ ($k = +1$):

$$a(\eta) \;=\; \frac{\Omega_m}{2(\Omega_m-1)}\left[1 \;-\; \cos\left(\sqrt{\Omega_m - 1}\,H_0\,\eta\right)\right]\;.$$

  This represents an **oscillating universe**. The period of oscillation is given by the conformal time

$$\eta = 2\pi H_0^{-1}(\Omega_m - 1)^{-1/2}\;.$$

– Cosmological Redschift

For sufficiently small distances, **Hubble's law** relates the distance d of a galaxy from us to its velocity v:

$$v \;=\; H_0\, d\;.$$

The velocity v can be measured via the Doppler-shift effect: $v = \Delta\lambda/\lambda$. A more accurate determination may be obtained via the phenomenon of the cosmological redshift.

Suppose an observer from a galaxy of a comoving distance R emits at time t_e a series of light pulses of period δt_e, which are observed on earth at time t_0 with a period δt_0. Because all pulses travel the same R, it then holds

$$R \;=\; \int\limits_{t_e}^{t_0} \frac{dt}{a(t)} \;=\; \int\limits_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)} \quad\Rightarrow\quad \frac{\delta t_0}{a(t_0)} \;=\; \frac{\delta t_e}{a(t_e)}\;.$$

Since it is $\nu_{e,0} = 1/(\delta t_{e,0})$ for the emitted and observed frequencies of light, the following formula for the **cosmological redshift** is obtained:

$$\frac{\nu_e}{\nu_0} \;=\; \frac{\lambda_0}{\lambda_e} \;=\; \frac{a(t_0)}{a(t_e)} \;=\; 1 \,+\, z\;,$$

where astronomers call z the **cosmological redshift** or simply **redshift**. The most distant quasar has a redshift $z = 6.6$.

– Cosmological Distance Measures

In addition to the cosmological redshift, the geometry of an expanding FRW universe affects other observables used to measure cosmological distances as well, such as the observed energy flux of light.

The energy flux f of photons received on earth from a remote source of luminosity L is given by

$$f \;=\; \frac{L}{4\pi d_{\rm eff}^2}\,\frac{1}{(1+z)^2}\,,$$

where $d_{\rm eff}$ is an *effective* distance between source and observer, which is defined such that the area of the sphere is $4\pi d_{\rm eff}^2$. This is *not* our true distance from the source, unless space is flat (*Why?*).

In the above formula for f, the first factor $L/(4\pi d_{\rm eff}^2)$ is the naive geometric law of flux reduction by the inverse distance squared.

The second factor $1/(1+z)^2$ is the result of 2 facts:

(i) the photons arrive with lower energy E_0 than the one emitted E_e, i.e. $E_0 = E_e/(1+z)$.

(ii) the photons are received less frequently compared to the source's emission time, because $\delta t_0 = (1+z)\,\delta t_e$.

– The Flatness Problem

Prediction for a radiation-dominated FRW universe:

$$\frac{|\Omega_0 - 1|}{|\Omega_r - 1|} \;=\; \frac{t_0}{t_r} \;=\; \frac{10^{17}\ {\rm sec}}{10^{-43}\ {\rm sec}} \;=\; 10^{60}$$

Degree of tuning required: 1 part in 10^{60}!

Solution to the Flatness Problem through Inflation:

$$\frac{|\Omega_r - 1|}{|\Omega_i - 1|} \;\approx\; \frac{a_i^2}{a_r^2} \;\approx\; e^{-2H(t_r - t_i)} \;\lesssim\; 10^{-60}$$

$$\Longrightarrow\; \mathcal{N}_e \;\approx\; H(t_r - t_i) \;\gtrsim\; 60$$

One needs a sufficient long period of inflation of ~ 60 e-folds.

Other problems solved by inflation: horizon and homogeneity problems, dilution of unwanted relics and defects etc.