Lectures on SUperSYmmetry

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8. SUperGRAvity

- Local Supersymmetry
- Non-renormalizable Interactions and Kähler Potential
- Gravity-Mediated SUSY Breaking

1. Introduction: Why SUSY?

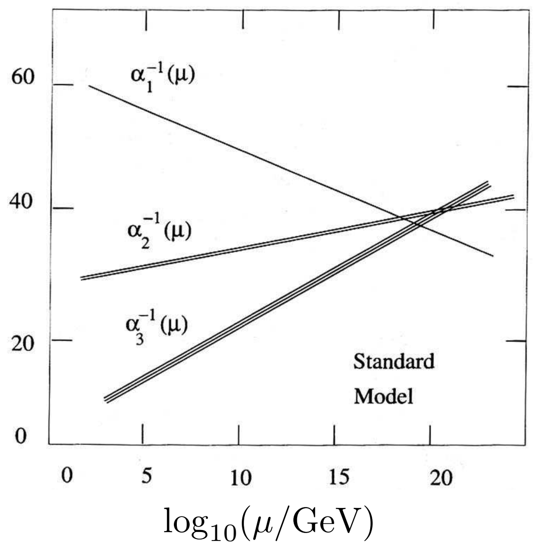
1. Electromagnetism → Quantum ElectroDynamics:
$U(1)_{\rm em}$
Force carrier: photon, γ, massless, spin $= 1\hbar$
Coupling to charged matter particles, such as e, u, d quarks.
Strength of the coupling $\alpha_{\rm em}(m_e) = 1/137$.

2. Weak interactions → Quantum WeakDynamics:
$SU(2)_L \otimes U(1)_Y / U(1)_{\rm em}$
Force carriers: W^+, W^-, Z bosons, massive, spin $= 1\hbar$
Coupling to particles with weak charges.
Strength of the coupling $\alpha_w(M_Z) \approx 1/30$.
Observed weakness due to the massiveness of $W^\pm$ and Z: $M_W, M_Z \sim 100$ GeV.

3. Strong interactions → Quantum ChromoDynamics:
$SU(3)_{\rm color}$
Force carriers: 8 massless gluons, g^a, spin $= 1\hbar$
Coupling to coloured particles, such as u, d quarks.
Strength of the coupling $\alpha_s(M_Z) \approx 1/10$.

4. Gravity → Quantum Gravity (?):
No known self-consistent quantum theory: Superstrings, large groups (E_6, etc.), extra dims. (?).
Force carrier: massless gravitons, with spin $= 2\hbar$.
$\sim 10^{-40}$ weaker than Electromagnetism.

The Standard Model: $SU(3)_{\rm color} \otimes SU(2)_L \otimes U(1)_Y$

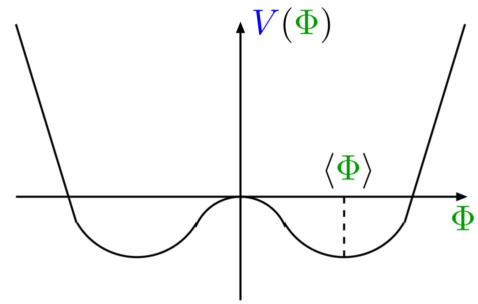


Important questions:

(i) What is the mechanism for giving masses to $W^\pm$, Z bosons and matter?

(ii) What guarantees stability of masses under quantum-mechanical effects from M_Z to $M_{\rm U} \sim 10^{16}$ GeV? (The so-called Gauge-Hierarchy problem)

(i) The Higgs mechanism



Higgs potential $V(\Phi)$ of a scalar field Φ (spin $= 0$):

$$V(\Phi) \;=\; -m^2\Phi^\dagger\Phi \;+\; \lambda(\Phi^\dagger\Phi)^2$$

is symmetric under $SU(2)_L \otimes U(1)_Y$, but not the ground state

$$\langle\Phi\rangle \;=\; \sqrt{\frac{m^2}{2\lambda}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which carries a weak charge, but no electric charge and colour. $W^\pm$, Z & matter feel the presence of $\langle\Phi\rangle$, but not γ and g^a

After Spontaneous Symmetry Breaking:

⇒ $W^\pm$, Z bosons and matter become massive, but not γ and g^a, e.g. $M_W = g_w\langle\Phi\rangle$

⇒ Quantum excitations of $\Phi = \langle\Phi\rangle \;+\; H \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
H is the so-called Higgs boson; spin $= 0$.

(ii) SUperSYmmetry introduces a new quantum dimension

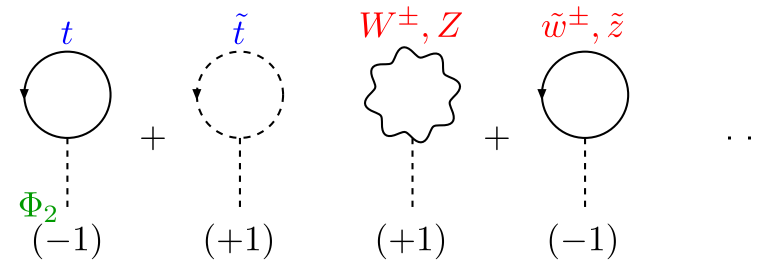
$\Rightarrow$ doubling of the particle spectrum of the SM:

Matter particles, spin $= 1/2$	$\Rightarrow$	SUSY-partners, spin $= 0$
e^-, μ^-, u, d, . . . , t		$\tilde{e}$, $\tilde{\mu}$, $\tilde{u}$, $\tilde{d}$, . . . , $\tilde{t}$
Anti-Matter, spin $= 1/2$	$\Rightarrow$	SUSY-partners, spin $= 0$
e^+, μ^+, $\bar{u}$, $\bar{d}$, . . . , $\bar{t}$		$\tilde{e}^*$, $\tilde{\mu}^*$, $\tilde{u}^*$, $\tilde{d}^*$, . . . , $\tilde{t}^*$
Force carriers, spin $= 1$	$\Rightarrow$	SUSY-partners, spin $= 1/2$
γ, W^+, W^-, Z, g		$\tilde{\gamma}$, $\tilde{w}^+$, $\tilde{w}^-$, $\tilde{z}$, $\tilde{g}$
Higgs bosons, spin $= 0$	$\Rightarrow$	SUSY-partners, spin $= 1/2$
2 Higgs doublets: Φ_1, Φ_2		$\tilde{h}_1^0$, $\tilde{h}_1^+$, $\tilde{h}_2^0$, $\tilde{h}_2^+$

No SUSY-partners observed yet

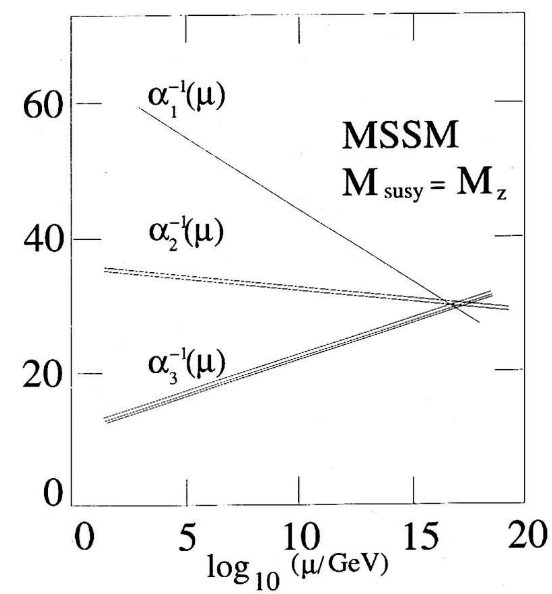
$\Rightarrow$ $\widetilde{\text{Mass}} - \text{Mass} = M_{\text{SUSY}} \gtrsim 100$ GeV.

Quantum fluctuations of the ground state:



$$= \begin{cases} 0, & \text{if SUSY is exact: } M_{\text{SUSY}} = 0 \\ (0.1 - 1)\ \text{TeV}^3, & \text{if SUSY is softly broken, with } M_{\text{SUSY}} = 1 \text{ TeV} \end{cases}$$

Accurate unification of couplings !



– Literature

Recommended Texts:

- J. Wess and J. Bagger, *Supersymmetry and Supergravity*, (Princeton University Press, Princeton NJ, 1992); Chapters: 1–8.
- D. Bailin and A. Love, *Supersymmetric Gauge Field Theory and String Theory*, (Institute of Physics Publishing, Bristol UK, 1994); Chapters: 4–6.

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Useful references:

- S.P. Martin, *A Supersymmetry Primer*, hep-ph/9709356.
- H.J.W. Müller-Kirsten and A. Wiedemann, *Supersymmetry: An Introduction with Conceptual and Calculational Details*, (World Scientific, Singapore, 1987).
- H.E. Haber and G.L. Kane, Phys. Rep. **117** (1985) 75; Appendices A–D.
- L.H. Ryder, *Quantum Field Theory*, (CUP, Cambridge UK, 1996) Second Edition.
- S. Weinberg, *Supersymmetry*, (CUP, Cambridge UK, 2000).
- P.C. West, *Introduction to Supersymmetry and Supergravity*, (World Scientific, Singapore, 1990).

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Prerequisites:

- H.F. Jones, *Groups, Representations and Physics*, (IOP, Oxford UK, 1998) Second Edition; Chapters: 2,3,6–11.
- A. Pilaftsis, Lecture notes PC4702 on *Symmetries in Physics*, `http://pilaftsi.home.cern.ch/pilaftsi/`; Chapters: 1–8.

– The Coleman–Mandula Theorem

Assumptions:

(i) S-matrix is based on a *local*, *relativistic*, *4-dimensional* Quantum Field Theory.

(ii) There are finite number of particles with mass m less than a given mass scale Λ.

(iii) Energy gap between vacuum and 1-particle states

(iv) Technical assumptions related to representation (rep) of operators and IR problems.

Consequences:

Then, the most general Lie algebra of symmetries of the S-matrix contains:

(i) the generators P_μ and $L_{\mu\nu}$ of the Poincaré group;

(ii) possible scalar operators B_l, i.e. $[B_l, P_\mu] = [B_l, L_{\mu\nu}] = 0$, which satisfy independently a Lie algebra L:

$$[B_l, B_m] \;=\; i f^k_{lm}\, B_k\,,$$

where f^k_{lm} are the structure constants of the algebra L.

– Supersymmetric transformations

$$Q|\mathrm{Boson}\rangle = |\mathrm{Fermion}\rangle\,, \qquad Q|\mathrm{Fermion}\rangle = |\mathrm{Boson}\rangle$$

Haag–Lopuszanski–Sohnius extension of the Coleman–Mandula theorem includes a graded Lie algebra, namely includes anti-commutators as well:

$$\begin{aligned}\{Q_\alpha,\ \bar{Q}_{\dot\alpha}\} &= 2\,(\sigma^\mu)_{\alpha\dot\alpha}\,P_\mu\,,\\ \{Q_\alpha,\ Q_\beta\} &= \{\bar{Q}_{\dot\alpha},\ \bar{Q}_{\dot\beta}\} = 0\,,\\ [Q_\alpha,\ P_\mu] &= [\bar{Q}_{\dot\alpha},\ P_\mu] = 0\,.\end{aligned}$$

Consequences:

- Equal number of (on-shell or off-shell) fermionic and bosonic degrees of freedom (dof).
- Scalar supermultiplet $\widehat{\Phi} \supset (\phi,\ \xi,\ F)$, where ϕ is a complex scalar (2 dof), ξ is a 2-component complex spinor (4 dof), and F is an auxiliary complex scalar (2 dof).
- Vector supermultiplet $\widehat{V}^a \supset (A^a_\mu,\ \lambda^a,\ D^a)$, where A^a_μ are massless (gauge-fixed) non-Abelian gauge fields (3 dof each), λ^a are the 2-component gauginos (4 dof each), and D^a are the auxiliary real fields (1 dof each).

– Dirac and Majorana fermions

I. *Dirac fermions in the chiral representation*

$$\mathcal{L}_{\mathrm{Dirac}} = \overline{\Psi}_D\, i\gamma^\mu\partial_\mu\Psi_D - m_D\overline{\Psi}_D\Psi_D\,,$$

where

$$\Psi_D = \begin{pmatrix}\xi_\alpha\\ \bar\eta^{\dot\alpha}\end{pmatrix}, \qquad \gamma_\mu = \begin{pmatrix}0 & (\sigma_\mu)_{\alpha\dot\beta}\\ (\bar\sigma_\mu)^{\dot\alpha\beta} & 0\end{pmatrix}$$

and $\overline{\Psi}_D = (\eta^\alpha\,,\ \bar\xi_{\dot\alpha})$, with $\sigma^\mu = (\mathbf{1}_2,\ \boldsymbol{\sigma})$ and $\bar\sigma^\mu = (\mathbf{1}_2,\ -\boldsymbol{\sigma})$. Note that $\boldsymbol{\sigma}$ denote the Pauli matrices:

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Our metric convention is $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

Chirality projection operators:

$$P_{L,R} = \frac{1}{2}\,(\mathbf{1}_4 \pm \gamma_5)\,, \qquad \gamma_5 = \begin{pmatrix}\delta_\alpha^\beta & 0\\ 0 & -\delta^{\dot\alpha}_{\dot\beta}\end{pmatrix},$$

or equivalently $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

Chirality states:

$$P_L\,\Psi_D = \begin{pmatrix}\xi_\alpha\\ 0\end{pmatrix}, \qquad P_R\,\Psi_D = \begin{pmatrix}0\\ \bar\eta^{\dot\alpha}\end{pmatrix}.$$

II. *Lorentz transformation properties of the Weyl spinors*

The Dirac spinor ψ_D consists of two Weyl spinors ξ_α and $\bar{\eta}^{\dot{\alpha}}$ that transform under the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ reps of the Lorentz group SO$(1,3) \simeq$ SL$(2,\mathbb{C})$.

The Lorentz trans properties of the Weyl spinors are:

$$\begin{aligned} \xi'_\alpha &= M_\alpha{}^\beta \xi_\beta\,, & \bar{\eta}'_{\dot{\alpha}} &= M^{\dagger}{}^{\dot{\beta}}{}_{\dot{\alpha}} \bar{\eta}_{\dot{\beta}}\,, \\ \xi'^\alpha &= M^{-1}_\beta{}^\alpha \xi^\beta\,, & \bar{\eta}'^{\dot{\alpha}} &= M^{\dagger -1}{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\eta}^{\dot{\beta}}\,. \end{aligned}$$

with $M \in \mathrm{SL}(2,\mathbb{C})$.

Duality relations among 2-spinors:

$$(\xi^\alpha)^\dagger = \bar{\xi}^{\dot{\alpha}}\,, \quad (\xi_\alpha)^\dagger = \bar{\xi}_{\dot{\alpha}}\,, \quad (\bar{\eta}_{\dot{\alpha}})^\dagger = \eta_\alpha\,, \quad (\eta^\alpha)^\dagger = \bar{\eta}^{\dot{\alpha}}$$

Lowering and raising spinor indices:

$$\xi_\alpha = \varepsilon_{\alpha\beta}\xi^\beta\,, \quad \xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta\,, \quad \bar{\eta}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\eta}^{\dot{\beta}}\,, \quad \bar{\eta}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\eta}_{\dot{\beta}}\,,$$

with $\varepsilon^{\alpha\beta} \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}} \equiv i\sigma_2 = -\varepsilon_{\dot{\alpha}\dot{\beta}}$.

Lorentz–invariant spinor contractions:

$$\xi\eta \equiv \xi^\alpha\eta_\alpha = \xi^\alpha\varepsilon_{\alpha\beta}\eta^\beta = -\eta^\beta\varepsilon_{\alpha\beta}\xi^\alpha = \eta^\beta\varepsilon_{\beta\alpha}\xi^\alpha = \eta^\beta\xi_\beta = \eta\xi$$

Likewise, $\bar{\xi}\bar{\eta} \equiv (\eta\xi)^\dagger = \xi^\dagger_\alpha \eta^{\alpha\dagger} = \bar{\xi}_{\dot{\alpha}}\bar{\eta}^{\dot{\alpha}} = \bar{\eta}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} = \bar{\eta}\bar{\xi}$.

<u>*Exercises*</u>:

(i) Show that

$$\bar{\xi}\bar{\sigma}^\mu\eta \equiv \bar{\xi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\eta_\beta = -\,\eta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}} \equiv -\eta\sigma^\mu\bar{\xi}\,.$$

(ii) Use (i) to verify that up to a total derivative $\propto \partial_\mu(\bar{\eta}\bar{\sigma}^\mu\eta)$, we get

$$\begin{aligned} \mathcal{L}_{\mathrm{Dirac}} &= \overline{\Psi}_D\, i\gamma^\mu\partial_\mu\Psi_D - m_D\overline{\Psi}_D\Psi_D\,, \\ &= \bar{\xi}\, i\bar{\sigma}^\mu\partial_\mu\xi + \bar{\eta}\, i\bar{\sigma}^\mu\partial_\mu\eta - m_D\,(\xi\eta + \bar{\eta}\bar{\xi})\,. \end{aligned}$$

(iii) Show that

$$M\sigma_\mu M^\dagger = \Lambda^\nu{}_\mu\,\sigma_\nu \quad \text{and} \quad M^{\dagger -1}\bar{\sigma}_\mu M^{-1} = \Lambda^\nu{}_\mu\,\bar{\sigma}_\nu\,,$$

where $\Lambda^\mu{}_\nu$ $\in$ SO(1,3), i.e. $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

(iv) Use (iii) to show that $\mathcal{L}_{\mathrm{Dirac}}$ is invariant under Lorentz trans.

III. *Majorana fermions*

Charge conjugation operator: $C = -i\gamma^2\gamma^0 = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix}$

Charge-conjugate of a 4-spinor $\Psi = \begin{pmatrix} \xi_\alpha \\ \bar\eta^{\dot\alpha} \end{pmatrix}$:

$$\Psi^C \equiv C\bar\Psi^T = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \begin{pmatrix} \eta^\beta \\ \bar\xi_{\dot\beta} \end{pmatrix} = \begin{pmatrix} \eta_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}$$

Definition of Majorana spinor:

$$\Psi_M \equiv \Psi_M^C = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix} \quad \text{or} \quad \xi = \eta$$

Kinetic Lagrangian for a Majorana field:

$$\begin{aligned} \mathcal{L}_{\rm Majorana} &= \frac{1}{2}\overline{\Psi}_M\, i\gamma^\mu\partial_\mu\Psi_M \,-\, \frac{1}{2}m_M\overline{\Psi}_M\Psi_M\,, \\ &= \bar\xi\, i\bar\sigma^\mu\partial_\mu\xi \,-\, \frac{1}{2}m_M\,(\xi\xi+\bar\xi\bar\xi\,) \end{aligned}$$

Glossary:

I. *Dirac fermions*

$$\begin{aligned} \overline{\Psi}_1 P_L\Psi_2 &= \eta_1\xi_2\,, \quad \overline{\Psi}_1 P_R\Psi_2 \;=\; \bar\xi_1\bar\eta_2\,, \\ \overline{\Psi}_1\gamma_\mu P_L\Psi_2 &= \bar\xi_1\bar\sigma_\mu\xi_2\,, \quad \overline{\Psi}_1\gamma_\mu P_R\Psi_2 \;=\; -\,\bar\eta_2\bar\sigma_\mu\eta_1\,. \end{aligned}$$

II. *Majorana fermions*

$$\begin{aligned} \overline{\Psi}_1\Psi_2 &= \overline{\Psi}_2\Psi_1\,, \quad \overline{\Psi}_1\gamma_5\Psi_2 \;=\; \overline{\Psi}_2\gamma_5\Psi_1\,, \\ \overline{\Psi}_1\gamma_\mu\Psi_2 &= -\,\overline{\Psi}_2\gamma_\mu\Psi_1\,, \quad \overline{\Psi}_1\gamma_\mu\gamma_5\Psi_2 \;=\; \overline{\Psi}_2\gamma_\mu\gamma_5\Psi_1\,. \end{aligned}$$

2. The simplest SUSY model: the WZ Model

– Non-interacting WZ model

$$\begin{aligned}\mathcal{L}_{\rm kin} &= \mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm fermion} \\ &= (\partial^\mu \phi^\dagger)(\partial_\mu \phi) + \bar{\xi}\, i\bar{\sigma}^\mu(\partial_\mu \xi)\,; \quad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\end{aligned}$$

Consider $\phi \to \phi + \delta\phi$ and $\phi^\dagger \to \phi^\dagger + \delta\phi^\dagger$, where

$$\delta\phi = \theta\xi \quad \text{and} \quad \delta\phi^\dagger = (\theta\xi)^\dagger = \bar{\xi}\bar{\theta} = \bar{\theta}\bar{\xi}\,,$$

and θ is an infinitesimal anticommuting 2-spinor constant.

$$\begin{aligned}\Rightarrow \mathcal{L}_{\rm scalar} &\to \mathcal{L}_{\rm scalar} + \delta\mathcal{L}_{\rm scalar}\,, \\ \delta\mathcal{L}_{\rm scalar} &= \theta(\partial^\mu\phi^\dagger)(\partial_\mu\xi) + \bar{\theta}(\partial^\mu\bar{\xi})(\partial_\mu\phi)\end{aligned}$$

Try $\xi_\alpha \to \xi_\alpha + \delta\xi_\alpha$ and $\bar{\xi}_{\dot\alpha} \to \bar{\xi}_{\dot\alpha} + \delta\bar{\xi}_{\dot\alpha}$, with

$$\delta\xi_\alpha = -i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\phi \quad \text{and} \quad \delta\bar{\xi}_{\dot\alpha} = i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^\dagger$$

$$\begin{aligned}\Rightarrow \mathcal{L}_{\rm fermion} &\to \mathcal{L}_{\rm fermion} + \delta\mathcal{L}_{\rm fermion}\,, \\ \delta\mathcal{L}_{\rm fermion} &= -\theta\sigma^\nu\bar{\sigma}^\mu(\partial_\mu\xi)(\partial_\nu\phi^\dagger) + \bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\theta}(\partial_\mu\partial_\nu\phi) \\ &= \theta\sigma^\nu\bar{\sigma}^\mu\xi(\partial_\mu\partial_\nu\phi^\dagger) + \bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\theta}(\partial_\mu\partial_\nu\phi) \\ &\quad - \partial_\mu\left[\theta\sigma^\nu\bar{\sigma}^\mu\xi(\partial_\nu\phi^\dagger)\right]\end{aligned}$$

Write $\sigma^\nu\bar{\sigma}^\mu = \frac{1}{2}\{\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\} - \frac{1}{2}[\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu]$, and a similar expression for $\bar{\sigma}^\mu\sigma^\nu$.

Since $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, only the symmetric term in $\{\cdots\}$ will survive.

Using the result of the exercise below, we find

$$\begin{aligned}\delta\mathcal{L}_{\rm fermion} &= \theta\xi(\partial_\mu\partial^\mu\phi^\dagger) + \bar{\xi}\bar{\theta}(\partial_\mu\partial^\mu\phi) \\ &= -\theta(\partial_\mu\xi)(\partial^\mu\phi^\dagger) - \bar{\theta}(\partial_\mu\bar{\xi})(\partial^\mu\phi) \\ &\quad + \partial_\mu\left[\theta\xi(\partial^\mu\phi^\dagger) + \bar{\xi}\bar{\theta}(\partial^\mu\phi)\right]\end{aligned}$$

$$\Rightarrow \delta\mathcal{L} = \delta\mathcal{L}_{\rm scalar} + \delta\mathcal{L}_{\rm fermion} = 0\ !$$

Exercise: Show that

$$\begin{aligned}(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\nu)^{\dot\alpha\beta} + (\sigma^\nu)_{\alpha\dot\alpha}(\bar{\sigma}^\mu)^{\dot\alpha\beta} &= 2\eta^{\mu\nu}\delta_\alpha^{\ \beta}\,, \\ (\bar{\sigma}^\mu)^{\dot\alpha\beta}(\sigma^\nu)_{\beta\dot\beta} + (\bar{\sigma}^\nu)^{\dot\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} &= 2\eta^{\mu\nu}\delta^{\dot\alpha}_{\ \dot\beta}\,.\end{aligned}$$

But, we are not finished yet !

The difference of two successive SUSY trans. must be a symmetry of the Lagrangian as well, i.e. the SUSY algebra should close.

$$\begin{aligned}(\delta_{\theta_2}\delta_{\theta_1} - \delta_{\theta_1}\delta_{\theta_2})\phi &= -i(\theta_1\sigma^\mu\bar{\theta}_2 - \theta_2\sigma^\mu\bar{\theta}_1)\,\partial_\mu\phi \\ &\equiv i\epsilon^\mu P_\mu\phi \quad (\text{with } \epsilon^{\mu*} = \epsilon^\mu) \\ (\delta_{\theta_2}\delta_{\theta_1} - \delta_{\theta_1}\delta_{\theta_2})\xi_\alpha &= -i(\sigma^\mu\bar{\theta}_1)_\alpha\theta_2\partial_\mu\xi + i(\sigma^\mu\bar{\theta}_2)_\alpha\theta_1\partial_\mu\xi \\ &\overset{\text{Fierz}}{=} -i(\theta_1\sigma^\mu\bar{\theta}_2 - \theta_2\sigma^\mu\bar{\theta}_1)\,\partial_\mu\xi_\alpha \\ &\quad + \theta_{1\alpha}\bar{\theta}_2 i\bar{\sigma}^\mu\partial_\mu\xi \;-\; \theta_{2\alpha}\bar{\theta}_1 i\bar{\sigma}^\mu\partial_\mu\xi\end{aligned}$$

Only for on-shell fermions, $i\bar{\sigma}^\mu\partial_\mu\xi = 0$, the SUSY algebra closes.

Exercise: Prove the Fierz identity:

$$\chi_\alpha(\xi\eta) \;+\; \xi_\alpha(\eta\chi) \;+\; \eta_\alpha(\chi\xi) \;=\; 0\,,$$

where χ, ξ and η are Weyl spinors.

To close the SUSY algebra off-shell, we need an *auxiliary* complex scalar F (without kinetic term) and add

$$\mathcal{L}_F \;=\; F^\dagger F$$

to $\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}}$, with

$$\begin{aligned}\delta F &= -i\bar{\theta}\bar{\sigma}^\mu(\partial_\mu\xi)\,, & \delta F^\dagger &= i(\partial_\mu\bar{\xi})\bar{\sigma}^\mu\theta \\ \delta\xi_\alpha &= -i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\phi + \theta_\alpha F\,, & \delta\bar{\xi}_{\dot{\alpha}} &= i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\phi^\dagger + \bar{\theta}_{\dot{\alpha}}F^\dagger\end{aligned}$$

Dimensions of the fields: $[\phi] = 1$, $[\xi] = \frac{3}{2}$, $[F] = 2$, $[\theta] = -\frac{1}{2}$.

Exercise: Prove **(i)** that the Lagrangian

$$\mathcal{L}_{\text{kin}} = (\partial^\mu\phi^\dagger)(\partial_\mu\phi) \;+\; \bar{\xi}\, i\bar{\sigma}^\mu(\partial_\mu\xi) \;+\; F^\dagger F$$

is invariant under the off-shell SUSY trans:

$$\begin{aligned}\delta\phi &= \theta\xi\,, & \delta\phi^\dagger &= \bar{\theta}\bar{\xi} \\ \delta\xi_\alpha &= -i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\phi + \theta_\alpha F\,, & \delta\bar{\xi}_{\dot{\alpha}} &= i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\phi^\dagger + \bar{\theta}_{\dot{\alpha}}F^\dagger \\ \delta F &= -i\bar{\theta}\bar{\sigma}^\mu(\partial_\mu\xi)\,, & \delta F^\dagger &= i(\partial_\mu\bar{\xi})\bar{\sigma}^\mu\theta\end{aligned}$$

and **(ii)** that the SUSY algebra closes off-shell:

$$(\delta_{\theta_2}\delta_{\theta_1} - \delta_{\theta_1}\delta_{\theta_2})X \;=\; -i(\theta_1\sigma^\mu\bar{\theta}_2 - \theta_2\sigma^\mu\bar{\theta}_1)\,\partial_\mu X\,,$$

with $X = \phi,\ \phi^\dagger,\ \xi,\ \bar{\xi},\ F,\ F^\dagger$.

– The Interacting WZ model

$$\mathcal{L}_{\rm WZ} \;=\; \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm int}$$
$$= (\partial^\mu \phi^\dagger)(\partial_\mu \phi) + \bar{\xi}\, i\bar{\sigma}^\mu(\partial_\mu \xi) + F^\dagger F$$
$$-\frac{1}{2}W_{\phi\phi}\,\xi\xi + W_\phi\, F - \frac{1}{2}W^\dagger_{\phi\phi}\bar{\xi}\bar{\xi} + F^\dagger W^\dagger_\phi$$

where

$$W(\phi) \;=\; \frac{m}{2}\,\phi\phi + \frac{h}{6}\,\phi\phi\phi$$

is the so-called superpotential, and

$$W_\phi \;=\; \frac{\delta W}{\delta\phi} \;=\; m\phi + \frac{h}{2}\phi^2$$
$$W_{\phi\phi} \;=\; \frac{\delta^2 W}{\delta\phi\,\delta\phi} \;=\; m + h\phi$$

Exercise: Show that up to total derivatives,

$$\mathcal{L}_{\rm int} \;=\; -\frac{1}{2}W_{\phi\phi}\,\xi\xi + W_\phi\, F - \frac{1}{2}W^\dagger_{\phi\phi}\bar{\xi}\bar{\xi} + W^\dagger_\phi F^\dagger$$
$$= -\frac{1}{2}(m + h\phi)\xi\xi - \frac{1}{2}(m + h\phi^\dagger)\bar{\xi}\bar{\xi}$$
$$+ (m\phi + \frac{h}{2}\phi^2)F + (m\phi^\dagger + \frac{h}{2}\phi^{\dagger 2})F^\dagger$$

remains invariant under off-shell SUSY transformations.

– Feynman rules

Equation of motions for the auxiliary fields F and $F^\dagger$:

$$F \;=\; -W^\dagger_\phi\,, \qquad F^\dagger \;=\; -W_\phi\,,$$

Substituting the above into $\mathcal{L}_{\rm WZ}$, we get

$$\mathcal{L}_{\rm WZ} \;=\; (\partial^\mu \phi^\dagger)(\partial_\mu \phi) + \bar{\xi}\, i\bar{\sigma}^\mu(\partial_\mu \xi) - W_\phi W^\dagger_\phi$$
$$-\frac{1}{2}\left(W_{\phi\phi}\,\xi\xi + W^\dagger_{\phi\phi}\bar{\xi}\bar{\xi}\right)$$

and the real potential is

$$V \;=\; W_\phi W^\dagger_\phi \;=\; m^2\phi^\dagger\phi + \frac{mh}{2}(\phi^\dagger\phi^2 + \phi^{\dagger 2}\phi) + \frac{h^2}{4}(\phi^\dagger\phi)^2$$

Exercise: If $\Psi = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}$ is a Majorana 4-spinor, show that the Ψ-dependent part of the WZ Lagrangian can be written down as

$$\mathcal{L}_\Psi \;=\; \frac{1}{2}\overline{\Psi}\, i\gamma^\mu\partial_\mu\Psi - \frac{1}{2}m\,\overline{\Psi}\,\Psi$$
$$-\frac{h}{2}\phi\,\overline{\Psi}\,P_L\,\Psi - \frac{h}{2}\phi^\dagger\overline{\Psi}\,P_R\,\Psi$$

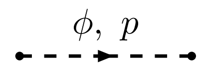
Summary

The complete WZ Lagrangian is

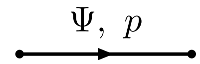
$$\mathcal{L}_{\rm WZ} \;=\; (\partial^\mu \phi^\dagger)(\partial_\mu \phi) \;-\; m^2 \phi^\dagger \phi \;+\; \frac{1}{2}\,\overline{\Psi}\, i\gamma^\mu \partial_\mu \Psi \;-\; \frac{1}{2}\, m\, \overline{\Psi}\,\Psi$$
$$\qquad -\,\frac{mh}{2}\,(\phi^\dagger \phi^2 + \phi^{\dagger 2}\phi) \;-\; \frac{h^2}{4}\,(\phi^\dagger \phi)^2$$
$$\qquad -\,\frac{h}{2}\,\phi\,\overline{\Psi}\,P_L\,\Psi \;-\; \frac{h}{2}\,\phi^\dagger \overline{\Psi}\,P_R\,\Psi\,,$$

where the F-field has been integrated out.

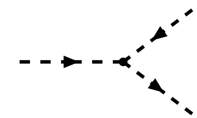
Feynman rules:



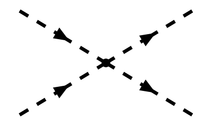
: $\dfrac{i}{p^2 - m^2}$



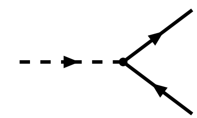
: $\dfrac{i}{\not{p} - m}$



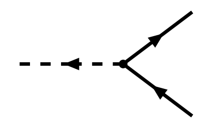
: $-imh$



: $-ih^2$



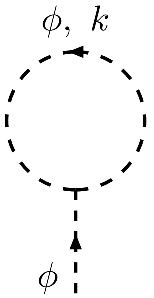
: $-ihP_L$



: $-ihP_R$

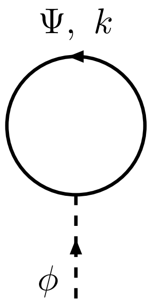
3. Non-renormalization Theorems in SUSY

– Absence of Tadpoles in the WZ Model



$$= \; (-imh) \int \frac{d^4k}{(2\pi)^4} \, \frac{i}{k^2 - m^2}$$

$+$



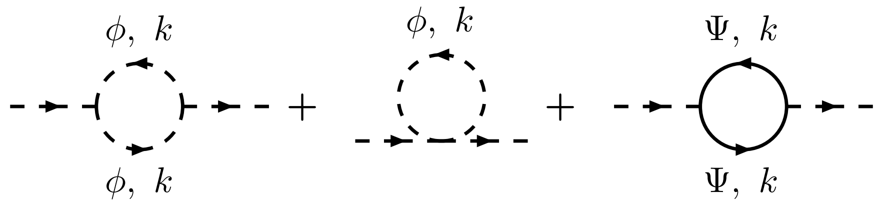
$$\begin{aligned} &= \; (-1)\, \tfrac{1}{2}\, (-ih) \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left(P_L \, \frac{i}{\not{k} - m} \right) \\ &= \; \tfrac{1}{2}\, (+ih) \int \frac{d^4k}{(2\pi)^4} \, \frac{i\mathrm{Tr}\,[P_L(\not{k} + m)]}{k^2 - m^2} \\ &= \; (+imh) \int \frac{d^4k}{(2\pi)^4} \, \frac{i}{k^2 - m^2} \end{aligned}$$

The sum of the two tadpole graphs is *exactly* zero !

If SUSY is exact, the vanishing of tadpoles holds to *all orders* of perturbation theory.

– Non-renormalization of Self-energy and Vertex interactions

$\phi\phi$–selfenergy at zero external momentum: $\Pi_{\phi\phi}(p^2 = 0)$



$$\begin{aligned} \Pi_{\phi\phi}(0) \; &= \; \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{(-imh)^2 i^2}{(k^2 - m^2)^2} \; + \; \frac{(-ih^2)\, i}{k^2 - m^2} \right. \\ &\quad \left. + \; (-1)\, \frac{1}{2}\, (-ih)^2 i^2 \, \frac{\mathrm{Tr}\,[P_L(\not{k} + m) P_R(\not{k} + m)]}{(k^2 - m^2)^2} \right\} \\ &= \; h^2 \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{m^2}{(k^2 - m^2)^2} \; + \; \frac{1}{k^2 - m^2} \right. \\ &\quad \left. - \; \frac{k^2}{(k^2 - m^2)^2} \right\} \; = \; 0 \end{aligned}$$

This result holds to all orders of perturbation theory.

Remark: For $p^2 \neq 0$, $\Pi_{\phi\phi}(p^2)$ is non-zero and ultra-violet (UV) divergent, due to the wave-function renormalization of ϕ.

Exercise: *The proper one-loop vertex interaction $\phi\phi\phi$*

1. Draw all Feynman graphs that contribute to the irreducible $\phi\phi\phi$ vertex $\Gamma^{\phi\phi\phi}$.

2. Identify the proper spin-statistics and combinatorial factors for the individual contributing graphs.

3. By choosing an appropriate routing for the loop momentum, show that the one-particle irreducible three-point correlation function $\Gamma^{\phi\phi\phi}$ vanishes identically in the limit of zero-momentum for the external ϕ particles.

4. Which other one-particle irreducible higher-point correlation functions do you expect to vanish in the zero-momentum limit for the external particles?

– Soft-SUSY Breaking

SUSY is not an exact symmetry of nature. For example, no scalar electron was ever observed or produced at colliders with a mass equal to that of the ordinary electron.

However, while breaking SUSY, we should not destroy all good quantum-mechanical properties as described by the non-renormalization theorems. Therefore, we should break SUSY softly, namely by adding to the Lagrangian terms of dimension less than 4, such as

$$\mathcal{L}_{\rm soft} = m_S^2\, \phi^\dagger\phi \;+\; \left(\frac{B_m m}{2}\,\phi\phi \;+\; \frac{hA_h}{6}\,\phi\phi\phi \;+\; {\rm h.c.}\right)$$

The remarkable feature of $\mathcal{L}_{\rm soft}$ is that it does guarantee the absence of quadratic UV divergences at all orders of perturbation theory.

How is $\mathcal{L}_{\rm soft}$ generated?

Several different mechanisms for generating SUSY breaking within string and supergravity models: [e.g., see textbook by S. Weinberg]

(i) Gravity-mediated SUSY breaking
(ii) Gauge-mediated SUSY breaking
(iii) Anomaly-mediated SUSY breaking
(iv) ⋮

Problem:

Consider the WZ model, in which SUSY is softly broken by the mass operator $-m_S^2\,\phi^\dagger\phi$.

1. What is the squared mass m_ϕ^2 of the ϕ-particle in this soft-SUSY broken WZ model? How much does m_ϕ now differ from the corresponding mass of the Majorana field Ψ?

2. Show that the $\phi\phi$ self-energy $\Pi_{\phi\phi}(0)$ in this extended WZ model not only does not vanish, but it is even infinite. (Hint: To evaluate the loop integral, you may use the substitution: $\int_{-\infty}^{+\infty} d^4k \to \pi^2 \int_{-\infty}^{0} k^2 dk^2$.)

3. *Absence of UV quadratic divergences*:
   Consider an UV cut-off regulator Λ^2, i.e. replace the integration limit $-\infty$ with $-\Lambda^2$ in the above loop integral, to show that $\Pi_{\phi\phi}(0)$ can only diverge as $\ln\Lambda^2$ as $\Lambda^2 \to \infty$, while all quadratic UV terms $\propto \Lambda^2$ cancel out.

4. *Technical solution to the gauge hierarchy problem*:
   If $\Lambda = M_{\rm Planck} = 10^{16}$ TeV represents a natural UV cut-off scale, calculate approximatively the maximally allowed value for m_S by requiring that $|\Pi_{\phi\phi}(0)|$ is smaller than m^2. For your calculations, you may assume $m = m_{\rm top}$ and $h = 1$.

4. Superfield Formulation of SUSY

Superfield formulation of SUSY is based on the superspace:

$$x^\mu\,, \quad \theta_\alpha\,, \quad \bar\theta_{\dot\alpha},$$

where $\theta_\alpha\,,\ \bar\theta_{\dot\alpha}$ are x-independent 2-component spinors.

– Generators of the Super-Poincaré Group

The generators super-Poincaré algebra are $P_\mu\,,\ J_{\mu\nu}\ \in\ L_0$ and the spinors $Q_\alpha\,,\ \bar Q_{\dot\alpha}\ \in\ L_1$. They satisfy the following $\mathbb{Z}_2$-graded Lie algebra:

$$
\begin{aligned}
\text{(i)}\quad & [P_\mu\,,\ P_\nu] \;=\; 0\,,\\
\text{(ii)}\quad & [P_\mu, J_{\rho\sigma}] \;=\; i\,(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho)\,,\\
\text{(iii)}\quad & [J_{\mu\nu}, J_{\rho\sigma}] \;=\; -i\,(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma})\,,\\
\text{(iv)}\quad & \{Q_\alpha,\ Q_\beta\} \;=\; \{\bar Q_{\dot\alpha},\ \bar Q_{\dot\beta}\} \;=\; 0\,,\\
\text{(v)}\quad & \{Q_\alpha,\ \bar Q_{\dot\beta}\} \;=\; 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,,\\
\text{(vi)}\quad & [Q_\alpha,\ P_\mu] \;=\; 0\,,\\
\text{(vii)}\quad & [J_{\mu\nu}\,,\ Q_\alpha] \;=\; -i(\sigma_{\mu\nu})_\alpha^{\ \beta}\,Q_\beta\,,\\
\text{(viii)}\quad & [J_{\mu\nu}\,,\ \bar Q_{\dot\alpha}] \;=\; -i(\bar\sigma_{\mu\nu})_{\dot\alpha}^{\ \dot\beta}\,\bar Q_{\dot\beta}\,,
\end{aligned}
$$

where $(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{1}{4}\,[\,(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} - (\sigma^\nu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\,]$ and $(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{1}{4}\,[\,(\bar\sigma^\mu)^{\dot\alpha\beta}(\sigma^\nu)_{\beta\dot\beta} - (\bar\sigma^\nu)^{\dot\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}\,]$.

Remarks:

- The commutation relations (i)–(iii) guarantee the Lorentz invariance (covariance) of the QFT.
- The anti-commutation relations (iv) and (v) have to do with the structure of the SUSY vacuum and their consequences will be discussed in Chapter 6.
- The commutation relations (vi)–(viii) imply that all members of a supermultiplet have the same mass and the number of fermionic and bosonic dof are equal. The latter will be made explicit in our construction of chiral and vector superfields.

...

Exercises:

(i) Verify that the differential operators:

$$P_\mu \;=\; i\partial_\mu\,,\;\; J_{\mu\nu} \;=\; x_\mu P_\nu - x_\nu P_\mu + i\theta^\alpha(\sigma_{\mu\nu})_\alpha^{\ \beta}\partial_\beta - i\bar\theta_{\dot\alpha}(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\bar\partial^{\dot\beta}\,,$$

$$Q_\alpha \;=\; \partial_\alpha + i(\sigma^\mu)_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\partial_\mu\,,\quad \bar Q_{\dot\alpha} \;=\; \bar\partial_{\dot\alpha} + i\theta^\beta\,(\sigma^\mu)_{\beta\dot\alpha}\,\partial_\mu\,,$$

satisfy the super-Poincaré algebra, where $\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$, $\bar\partial_{\dot\alpha} \equiv \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$, $\partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha} = -\varepsilon^{\alpha\beta}\,\partial_\beta$ and $\bar\partial^{\dot\alpha} \equiv \frac{\partial}{\partial\bar\theta_{\dot\alpha}} = -\varepsilon^{\dot\alpha\dot\beta}\,\partial_{\dot\beta}$.

(ii) Prove the $\mathbb{Z}_2$-graded Jacobi identity:

$$[J_{\mu\nu}\,,\ \{Q_\alpha\,,\ \bar Q_{\dot\beta}\}] + \{Q_\alpha\,,\ [Q_{\dot\beta}\,,\ J_{\mu\nu}]\} + \{\bar Q_{\dot\beta}\,,\ [Q_\alpha\,,\ J_{\mu\nu}]\} = 0\,.$$

– Chiral Superfields

Complex scalar field in superspace ($\equiv$ scalar superfield):

$$\begin{aligned}\Phi(x,\theta,\bar\theta) \;&=\; \phi(x) \,+\, \theta\xi(x) \,+\, \bar\theta\bar\chi(x) \,+\, \theta^2 f(x)\\ &\quad +\,\bar\theta^2 g(x) \,+\, \theta\sigma^\mu\bar\theta\, V_\mu(x) \,+\, \theta^2\bar\theta\bar\lambda(x)\\ &\quad +\,\bar\theta^2\theta\eta(x) \,+\, \theta^2\bar\theta^2 d(x)\,.\end{aligned}$$

Note that $\Phi(x,\theta,\bar\theta)$ contains 4 complex scalars (8 dof), 4 complex Weyl spinors (16 dof) and one Lorentz vector (4 dof), implying that the bosonic and fermionic dofs are not equal.

Hence, without further constraints, $\Phi(x,\theta,\bar\theta)$ cannot be an irreducible representation (irrep) of SUSY.

...

Exercise: Prove the following identities:

$$\text{(i)}\qquad \theta_\alpha\,\theta_\beta \;=\; \frac{1}{2}\,\varepsilon_{\alpha\beta}\,\theta^2\,,$$

$$\text{(ii)}\qquad \bar\theta_{\dot\alpha}\,\bar\theta_{\dot\beta} \;=\; -\frac{1}{2}\,\varepsilon_{\dot\alpha\dot\beta}\,\bar\theta^2\,,$$

$$\text{(iii)}\qquad (\theta\sigma^\mu\bar\theta)\,(\theta\sigma^\nu\bar\theta) \;=\; \frac{1}{2}\,\eta^{\mu\nu}\,\theta^2\,\bar\theta^2\,.$$

Guesswork:

To eliminate the many 'unbalanced' components in $\Phi(x,\theta,\bar\theta)$, we first try to impose the constraint:

$$\bar\partial_{\dot\alpha}\,\Phi(x,\theta,\bar\theta) \;=\; 0\,.$$

The above leaves intact the two scalars, $\phi(x)$ and $f(x)$ (4 bosonic dof), and the Weyl spinor $\xi(x)$ (4 fermionic dof). It looks a perfect guess!

But, this constraint is not maintained by a general super-Poincaré trans:

$$\bar\partial_{\dot\alpha}\,[e^{i(a^\mu P_\mu\,+\,\omega^{\mu\nu}J_{\mu\nu}\,+\,\zeta Q\,+\,\bar\zeta\bar Q)}\,\Phi(x,\theta,\bar\theta)] \;\neq\; 0\,.$$

Demanding that the validity of the constraint $\bar\partial_{\dot\alpha}\,\Phi = 0$ holds for an infinitesimal SUSY trans is equivalent to requiring that

$$[\,\bar\partial_{\dot\alpha},\; a^\mu P_\mu \;+\; \omega^{\mu\nu}J_{\mu\nu} \;+\; \zeta Q \;+\; \bar\zeta\bar Q\,] \;\propto\; 0 \text{ or } \bar\partial_{\dot\alpha}\,, \qquad (\star)$$

which is not satisfied, because

$$[\,\bar\partial_{\dot\alpha},\; \zeta Q\,] \;=\; -\,i(\zeta\sigma^\mu)_{\dot\alpha}\,\partial_\mu \;\not\propto\; \bar\partial_{\dot\alpha}\,.$$

A better choice would be to have a sort of "covariant derivative" $\bar D_{\dot\alpha}$ with respect to (w.r.t.) supertranslations:

$$[\,\bar D_{\dot\alpha},\; \zeta Q\,] \;=\; [\,\bar D_{\dot\alpha},\; \bar\zeta\bar Q\,] \;=\; 0\,,$$

and likewise for $\partial_\alpha \to D_\alpha$.

With little a bit of effort, we find that

$$D_\alpha \;=\; \partial_\alpha \;-\; i(\sigma^\mu)_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\partial_\mu\,, \quad \bar D_{\dot\alpha} \;=\; \bar\partial_{\dot\alpha} - i\theta^\beta\,(\sigma^\mu)_{\beta\dot\alpha}\,\partial_\mu$$

have the desired properties.

...

Exercises:

(i) Show that

$$\{D_\alpha,\; Q_\beta\} = \{D_\alpha,\; \bar Q_{\dot\beta}\} = \{\bar D_{\dot\alpha},\; Q_\beta\} = \{\bar D_{\dot\alpha},\; \bar Q_{\dot\beta}\} = 0\,.$$

(ii) Show that D_α and $\bar D_{\dot\alpha}$ do satisfy the general $(\star)$ condition on page 33.

Chiral Superfield. This is a scalar superfield that satisfies the constraint:

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) \;=\; 0 \qquad \text{(chiral)}$$
$$\underline{\text{or}} \qquad D_{\alpha}\Phi(x,\theta,\bar{\theta}) \;=\; 0 \qquad \text{(anti-chiral)}.$$

For example, any superfield $\Phi(y,\theta)$, with

$$y^\mu \;=\; x^\mu \;-\; i\theta\sigma^\mu\bar{\theta}\,,$$

but otherwise independent of $\bar{\theta}$, satisfies $\bar{D}_{\dot{\alpha}}\Phi = 0$.

Field expansion of $\Phi(y,\theta)$ for the case $\bar{D}_{\dot{\alpha}}\Phi = 0$:

$$\Phi(y,\theta) \;=\; \phi(y) \;+\; \sqrt{2}\,\theta\xi(y) \;+\; \theta^2 F(y)\,,$$

where the inserted $\sqrt{2}$ is just a convention.

$\Phi(y,\theta)$ contains the (complex) scalar ϕ (2 bosonic dofs), the (complex) auxiliary scalar F (2 bosonic dofs) and the complex Weyl spinor ξ (4 dofs):

$$2+2 \text{ bosonic dofs} = 4 \text{ fermionic dofs},$$

as it should be for an off-shell chiral supermultiplet.

Exercise: Verify that $\bar{D}_{\dot{\alpha}}\,\Phi(y,\theta) = 0\,$, by proving first that $\bar{D}_{\dot{\alpha}}\,\theta_\beta = 0$ and $\bar{D}_{\dot{\alpha}}\,y^\mu = 0\,$.

Field components of the chiral superfield:

$$\phi(x) \;=\; \Phi(y,\theta)|_{\theta,\bar{\theta}=0}\,,$$
$$\xi_\alpha(x) \;=\; \frac{1}{\sqrt{2}}\,D_\alpha\Phi(y,\theta)|_{\theta,\bar{\theta}=0}\,,$$
$$F(x) \;=\; \frac{1}{4}\,D^2\,\Phi(y,\theta)|_{\theta,\bar{\theta}=0}\,.$$

With the help of the relations:

$$Q_\alpha\,|_{\theta,\bar{\theta}=0} \;=\; D_\alpha + 2i(\sigma^\mu\bar{\theta})_\alpha\,\partial_\mu\,|_{\theta,\bar{\theta}=0}\,,$$
$$\bar{Q}_{\dot{\alpha}}\,|_{\theta,\bar{\theta}=0} \;=\; \bar{D}_{\dot{\alpha}} + 2i(\theta\sigma^\mu)_{\dot{\alpha}}\,\partial_\mu\,|_{\theta,\bar{\theta}=0}\,,$$

we can now find the SUSY trans of the component fields.

For example, the scalar component $\phi(x)$ transforms as

$$\delta_\zeta\phi \;=\; (\zeta Q \;+\; \bar{\zeta}\bar{Q})\,\Phi\,|_{\theta,\bar{\theta}=0} \;=\; (\zeta D \;+\; \underbrace{\bar{\zeta}\bar{D}}_{=0;\ \bar{D}\Phi=0}\;)\,\Phi\,|_{\theta,\bar{\theta}=0}$$
$$=\; \sqrt{2}\,\zeta\xi\,.$$

This is identical to the SUSY trans we found on page 17 using component fields only, after replacing $\sqrt{2}\,\zeta \to \theta$.

Exercises:

(i) Prove the following identities:

$$\partial_\alpha \theta^2 = 2\theta_\alpha\,, \qquad \bar\partial_{\dot\alpha}\bar\theta^2 = -\,2\,\bar\theta_{\dot\alpha}\,,$$
$$(\partial^\alpha \partial_\alpha)\,\theta^2 = 4\,, \quad (\partial_{\dot\alpha}\partial^{\dot\alpha})\,\bar\theta^2 = 4\,.$$

(ii) Use the field-component projections on page 36 and the relations of (i) above to show that

$$\delta_\zeta \xi_\alpha = \sqrt{2}\,\zeta_\alpha F - i\sqrt{2}\,(\sigma^\mu\bar\zeta)_\alpha\,\partial_\mu\phi\,,$$
$$\delta_\zeta F = i\sqrt{2}\,\partial_\mu(\xi\sigma^\mu\bar\zeta) = -i\sqrt{2}\,\partial_\mu(\bar\zeta\bar\sigma^\mu\xi)\,.$$

Observe that with $\sqrt{2}\,\zeta \to \theta$, the SUSY trans of the component fields become identical to those given in the exercise on page 20.

Remark: The above exercise tells us that the F component of a chiral superfield transforms into a total derivative under a SUSY trans.

– The WZ Model in terms of Superfields

An action $S = \int d^4x\,\mathcal{L}$ is invariant under a global (or local) trans if the Lagrangian $\mathcal{L}$ remains invariant up to a total derivative, e.g.

$$\mathcal{L} \stackrel{\text{SUSY}}{\to} \mathcal{L}' = \mathcal{L} + \partial_\mu Z^\mu\,,$$

for an arbitrary function Z^μ that vanishes at $x \to \pm\infty$.

Since the F-component of a chiral superfield $\Phi(y,\theta)$ transforms into a total derivative under SUSY, it can be used to build up SUSY invariant actions:

$$\int d^4x\;\Phi(y,\theta)\,|_{\theta^2} = \frac{1}{4}\int d^4x\;D^2\Phi(y,\theta)\,|_{\theta,\bar\theta=0}\,.$$

However, this term by itself gives only a linear term in F that on-shell vanishes.

We now notice that the product of two or more chiral superfields is also chiral:

$$\bar D_{\dot\alpha}\,\Phi^2 = (\bar D_{\dot\alpha}\,\Phi)\,\Phi + \Phi\,(\bar D_{\dot\alpha}\,\Phi) = 0\,,$$

because $\bar D_{\dot\alpha}$ is a linear differential operator and $\bar D_{\dot\alpha}\,\Phi = 0$.

Exercise: Show that if $\Phi(y,\theta)$ is a chiral superfield obeying $\bar D_{\dot\alpha}\,\Phi = 0$, then $\Phi^\dagger(y^\dagger,\bar\theta)$ is anti-chiral: $D_\alpha\,\Phi^\dagger = 0$.

A more general form of a SUSY invariant action can be constructed by means of a *chiral* polynomial $W(\Phi)$ in Φ:

$$W(\Phi) \;=\; t_F \Phi \;+\; \frac{1}{2} m^2 \Phi^2 \;+\; \frac{1}{6} h\, \Phi^3 \;+\; \cdots$$

$W(\Phi)$ is also called superpotential and is related to the one discussed on page 21.

With the help of the chiral superpotential W and its hermitean conjugate antichiral one $W^\dagger$, we can write down the SUSY-invariant action

$$\begin{aligned} S_W \;&=\; \int d^4x \,(\, W|_{\theta^2} \;+\; W^\dagger|_{\bar\theta^2}\,) \\ &=\; \frac{1}{4} \int d^4x \,(\, D^2 W \;+\; \bar D^2 W^\dagger\,)|_{\theta,\bar\theta=0}\,. \end{aligned}$$

However, S_W does *not* contain kinetic terms.

...

Exercise: Expand the chiral superfield $\Phi(y,\theta)$ in terms of x^μ, θ and $\bar\theta$:

$$\begin{aligned} \Phi(y,\theta) \;&=\; \phi(x) \;-\; i(\theta\sigma^\mu\bar\theta)\,\partial_\mu\phi(x) \;-\; \frac{1}{4}\theta^2\bar\theta^2\,\partial_\mu\partial^\mu\phi(x) \\ &\quad +\sqrt{2}\,\theta\xi \;+\; \frac{i}{\sqrt{2}}\theta^2\,(\partial_\mu\xi\,\sigma^\mu\bar\theta) \;+\; \theta^2\,F(x)\,. \end{aligned}$$

Kinetic terms for chiral superfields:

It can be shown that, up to a total derivative, the $\theta^2\bar\theta^2$–component $d(x)$ of an unconstrained superfield $\Phi(x,\theta,\bar\theta)$, remains invariant under a SUSY trans.

This implies that the $\theta^2\bar\theta^2$–component of the manifestly real superfield $\Phi^\dagger\Phi$ is SUSY invariant.

Hence, the SUSY-invariant kinetic action is

$$S_{\rm kin} \;=\; \int d^4x\,\Phi^\dagger\Phi\,|_{\theta^2\bar\theta^2} \;=\; \frac{1}{16}\int d^4x\, D^2\bar D^2\,\Phi^\dagger\Phi\,|_{\theta,\bar\theta=0}\,,$$

because $D^2\bar D^2\bar\theta^2\theta^2 = 16$.

...

Exercises:

(i) Show that up to a total derivative, the $\theta^2\bar\theta^2$–component $d(x)$ of a general Φ is invariant under SUSY trans:

$$\delta_\zeta d \;=\; \frac{i}{2}\partial_\mu\,(\,\xi\sigma^\mu\bar\zeta \;+\; \zeta\sigma^\mu\bar\lambda\,)\,,$$

where ξ and $\bar\lambda$ are Weyl spinors defined on page 32.

(ii) Calculate the $\theta^2\bar\theta^2$–component of $\Phi^\dagger\Phi$ to find that up to total derivatives,

$$\Phi^\dagger\Phi\,|_{\theta^2\bar\theta^2} \;=\; (\partial^\mu\phi^\dagger)(\partial_\mu\phi) + \bar\xi\, i\bar\sigma^\mu(\partial_\mu\xi) + F^\dagger F \;=\; \mathcal{L}_{\rm kin}\,,$$

where $\mathcal{L}_{\rm kin}$ is WZ kinetic Lagrangian given on page 20.

Summary:

The total SUSY-invariant action $S_{\rm tot}$ for one chiral superfield $\Phi(y,\theta)$ is

$$\begin{aligned} S_{\rm tot} &= S_{\rm kin} + S_W \\ &= \int d^4x \; (\Phi^\dagger\Phi\,|_{\theta^2\bar\theta^2} + W\,|_{\theta^2} + W^\dagger\,|_{\bar\theta^2}) \\ &= \int d^4x \; (\frac{1}{16} D^2\bar D^2\,\Phi^\dagger\Phi + \frac{1}{4} D^2 W + \frac{1}{4}\bar D^2 W^\dagger)\,|_{\theta,\bar\theta=0}, \end{aligned}$$

with $W(\Phi) = t_F\Phi + \frac{1}{2} m^2\Phi^2 + \frac{1}{6} h\,\Phi^3 + \cdots$

...

Exercises:

(i) Verify that

$$W(\Phi)\,|_{\theta^2} = W_\phi\, F - \frac{1}{2} W_{\phi\phi}\,\xi\xi\,,$$

where $W_\phi = \frac{\delta W(\Phi)}{\delta\Phi}|_{\Phi=\phi}$ and $W_{\phi\phi} = \frac{\delta^2 W(\Phi)}{\delta\Phi\delta\Phi}|_{\Phi=\phi}$. Convince yourself that the above result is consistent with the one presented for the WZ model on page 21.

(ii) Write down the renormalizable SUSY-invariant action of a model with two complex chiral multiplets Φ_1 and Φ_2, and calculate the *real* potential, after integrating out the corresponding auxiliary fields.

– Integration in Superspace

In addition to differentiation (e.g. $\partial_\beta\theta^\alpha = \delta^\alpha_\beta$; $\{\partial_\alpha,\partial_\beta\} = 0$), we may introduce the concept of integration over Grassman variables.

For *one* Grassman variable θ, integration over θ is defined as

$$\int d\theta\,\theta = 1\,, \qquad \int d\theta\, 1 = 0\,.$$

For the Grassman-valued function $f(x,\theta) = f_0(x) + f_1(x)\theta$, integrating over θ yields

$$\int d\theta\; f(x,\theta) = f_1(x) = \frac{\partial f(x,\theta)}{\partial\theta}\,.$$

$\therefore$ *Integration* is equivalent to *Differentiation* in superspace.

The superspace δ-function is defined by

$$\delta(\theta) = \theta\,,$$

which satisfies the known property:

$$\int d\theta\, f(x,\theta)\,\delta(\theta) = f(x,0) = f_0(x)\,.$$

Remark: The integral $\int d\theta$ is invariant under constant shifts of θ: $\int d\theta\; f(x,\theta+\zeta) = \int d\theta\; f(x,\theta)$.

The above concepts of integration may be extended to the superspace of $N = 1$ SUSY $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, by using the defining properties:

$$\int d^2\theta\, \theta^2 \;=\; \int d^2\bar{\theta}\, \bar{\theta}^2 \;=\; \int d^4\theta\, \theta^2\, \bar{\theta}^2 \;=\; 1\,,$$

with $d^4\theta \equiv d^2\theta\, d^2\bar{\theta}$.

...

Exercises:

(i) Check that the following definitions of the integration measures are consistent with the defining properties stated above:

$$d^2\theta \;\equiv\; -\frac{1}{4}\,\varepsilon_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta\,, \quad d^2\bar{\theta} \;\equiv\; -\frac{1}{4}\,\varepsilon^{\dot{\alpha}\dot{\beta}}\, d\bar{\theta}_{\dot{\alpha}}\, d\bar{\theta}_{\dot{\beta}}\,.$$

(ii) Show that $\delta(\theta) = \theta^2$ and $\delta(\bar{\theta}) = \bar{\theta}^2$ are the properly defined δ-functions in $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$–space.

(iii) Show that the total SUSY-invariant action can be written down as:

$$S_{\rm tot} \;=\; \int d^4x\, d^4\theta\, [\, \Phi^\dagger\Phi \,+\, W(\Phi)\,\delta(\bar{\theta}) \,+\, W^\dagger(\Phi^\dagger)\,\delta(\theta)\,]\,.$$

5. Supersymmetric Gauge Theories (SGTs)

– Vector Superfields

These are real superfields: $V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta})$, where

$$\begin{aligned} V(x,\theta,\bar{\theta}) \;=\;& C \,+\, \theta\chi \,+\, \bar{\theta}\bar{\chi} \,+\, \theta\sigma^\mu\bar{\theta}A_\mu \\ & +\frac{1}{2}\theta^2\,(M+iN) \,+\, \frac{1}{2}\bar{\theta}^2\,(M-iN) \\ & +\bar{\theta}^2\,\theta\,(\,\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}\,) \,+\, \theta^2\,\bar{\theta}\,(\,\bar{\lambda} - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi\,) \\ & +\frac{1}{2}\theta^2\,\bar{\theta}^2\,(D - \frac{1}{2}\partial_\mu\partial^\mu\, C\,)\,. \end{aligned}$$

8 bosonic dofs: C, D, M, N, V_μ;
8 fermionic dofs: χ, λ.
But, not all of them are physical dofs, and some of them can be 'gauged away'.

Local superfield redefinition (also called SUSY-guage trans):

$$V \;\to\; V' \;=\; V \,+\, i\,(\,\Lambda \,-\, \Lambda^\dagger) \qquad \textbf{(Abelian case)},$$

where the gauge parameter $\Lambda(y,\theta)$ $(\Lambda^\dagger(y^\dagger,\bar{\theta}))$ is a chiral (anti-chiral) superfield: $\bar{D}_{\dot{\alpha}}\Lambda = 0$ $(D_\alpha\Lambda^\dagger = 0)$.

Exercise: Starting from the general expression for V, perform a SUSY-gauge trans to show that the fields χ, C, M, N and one-component of the gauge field A_μ can be 'gauged away'.

The Wess–Zumino Gauge

The choice of SUSY-gauge fixing that eliminates $\chi,\ C,\ M,\ N$ and one-component of the gauge field A_μ is called the *Wess–Zumino* (WZ) gauge. In the WZ gauge, vector superfield V reads:

$$V_{\rm WZ}(x,\theta,\bar\theta) \;=\; \theta\sigma^\mu\bar\theta A_\mu \;+\; \bar\theta^2\,\theta\lambda \;+\; \theta^2\,\bar\theta\bar\lambda \;+\; \frac{1}{2}\theta^2\,\bar\theta^2\,D\,.$$

The WZ gauge breaks SUSY explicitly, but still allows the usual gauge trans (see exercise below).

Off-shell, $V_{\rm WZ}$ consists of the following field components:
the gauge-fixed vector field A_μ (3 dofs, $[A_\mu]=1$),
the auxiliary field D (1 dof, $[D]=2$),
and the Weyl spinor λ (4 dofs, $[\lambda]=3/2$).

On-shell, A_μ and λ have 2 dofs each.

Exercises:

(i) Verify that the special SUSY-gauge trans (Abelian case): $V_{\rm WZ} \;\to\; V'_{\rm WZ} \;=\; V_{\rm WZ} + \frac{i}{2}[\Lambda(y)-\Lambda(y^\dagger)]$, reproduces the usual gauge trans: $A_\mu(x) \;\to\; A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)$, where $y^\mu = x^\mu - i(\theta\sigma^\mu\bar\theta)$ and $\Lambda(x)=\Lambda^*(x)$.

(ii) Calculate the higher powers of $V_{\rm WZ}$ to obtain that

$$V^2_{\rm WZ} \;=\; \frac{1}{2}\,\theta^2\bar\theta^2\,A_\mu A^\mu\,, \quad V^{n\geq 3}_{\rm WZ} \;=\; 0\,.$$

Non-Abelian SUSY gauge transformations:

In non-Abelian theories, the vector superfield is defined in the adjoint rep:

$$V(x,\theta,\bar\theta) \;=\; V^a(x,\theta,\bar\theta)\,T^a\,,$$

where T^a are the generators of the gauge group, obeying the commutation relations (or the Lie algebra):

$$[\,T^a\,,\;T^b\,] \;=\; i\,f^{abc}\,T^c\,,$$

where f^{abc} are the so-called structure constants of the group.

The generators in the fundamental rep are normalized, such that ${\rm Tr}\,(T^aT^b) = \frac{1}{2}\delta^{ab}$.

Examples: $T^a_{\rm SU(2)} = \frac{1}{2}\sigma^a$ $(a=1,2,3)$ and $T^a_{\rm SU(3)} = \frac{1}{2}\lambda^a$ $(a=1,\dots,8)$, where σ^a and λ^a are the Pauli and Gell-Mann matrices, respectively. (*Question*: How many generators does an SU(N) group have?).

The important object for constructing actions is e^{2gV}, where g is the gauge coupling of the gauge group, e.g. SU(N).

A *general* non-Abelian SUSY gauge-trans is defined by

$$e^{2gV} \;\to\; e^{2gV'} \;=\; e^{-2ig\Lambda^\dagger}\,e^{2gV}\,e^{2ig\Lambda}\,,$$

where $\Lambda(y,\theta) = \Lambda^a(y,\theta)\,T^a$ is a chiral superfield, $\bar D_{\dot\alpha}\Lambda^a = 0$.

– Gauge sector of SGTs

The proper supersymmetric field strengths are

$$W_\alpha = W_\alpha^a T^a = -\frac{1}{8g}\bar{D}^2\,(\,e^{-2gV}\,D_\alpha\,e^{2gV}\,)\,,$$
$$\bar{W}_{\dot\alpha} = \bar{W}_{\dot\alpha}^a T^a = \frac{1}{8g}\bar{D}^2\,(\,e^{2gV}\,D_\alpha\,e^{-2gV}\,)\,.$$

These objects are chiral, i.e. $\bar{D}_{\dot\alpha}\,W_\alpha = D_\alpha\,\bar{W}_{\dot\alpha} = 0$, because $D^3 = \bar{D}^3 = 0$.

Most importantly, W_α and $\bar{W}_{\dot\alpha}$ transform gauge-covariantly:

$$W_\alpha \;\to\; e^{-2ig\Lambda}\,W_\alpha\,e^{2ig\Lambda}\,, \qquad \bar{W}_{\dot\alpha} \;\to\; e^{-2ig\Lambda^\dagger}\,\bar{W}_{\dot\alpha}\,e^{2ig\Lambda^\dagger}\,,$$

Proof:

Using the chiral properties of Λ and $\Lambda^\dagger$, $\bar{D}_{\dot\alpha}\Lambda = 0$ and $D_\alpha\Lambda^\dagger = 0$, we have

$$W_\alpha \to -\frac{1}{8g}\underbrace{\bar{D}^2\,[\,e^{-2ig\Lambda}}_{=e^{-2ig\Lambda}[\,\bar{D}^2}\,e^{-2gV}\;\underbrace{e^{2ig\Lambda^\dagger}D_\alpha}_{=D_\alpha\,e^{2ig\Lambda^\dagger}}(e^{-2ig\Lambda^\dagger}\,e^{2gV}\,e^{2ig\Lambda}\,)\,]$$
$$= -\frac{1}{8g}e^{-2ig\Lambda}\,\bar{D}^2\,[\,e^{-2gV}\,D_\alpha(e^{2gV}\,e^{2ig\Lambda}\,)\,]$$
$$= e^{-2ig\Lambda}\,W_\alpha\,e^{2ig\Lambda} - \frac{1}{8g}e^{-2ig\Lambda}\;\underbrace{\bar{D}^2\,D_\alpha}_{\bar{D}_{\dot\alpha}\,\{\bar{D}^{\dot\alpha},D_\alpha\}}\;e^{2ig\Lambda}\,.$$

We now use the fact that $\{\bar{D}_{\dot\alpha}, D_\alpha\} = -2(\sigma^\mu)_{\alpha\dot\alpha}\,P_\mu$ and $[\,\bar{D}_{\dot\alpha}, P_\mu\,] = 0$:

$$W_\alpha \to e^{-2ig\Lambda}\,W_\alpha\,e^{2ig\Lambda} - \frac{1}{8g}e^{-2ig\Lambda}\,\bar{D}_{\dot\alpha}\,\{\bar{D}^{\dot\alpha}, D_\alpha\}e^{2ig\Lambda}$$
$$= e^{-2ig\Lambda}\,W_\alpha\,e^{2ig\Lambda} + \frac{1}{8g}e^{-2ig\Lambda}\,(\sigma^\mu)_\alpha^{\;\dot\alpha}\,P_\mu\,\underbrace{\bar{D}_{\dot\alpha}\,e^{2ig\Lambda}}_{=\,0}\,,$$

q.e.d.

...

Exercises:

(i) Verify that $\{\bar{D}_{\dot\alpha}, D_\alpha\} = -2(\sigma^\mu)_{\alpha\dot\alpha}\,P_\mu$.

(ii) Calculate in the WZ gauge the spinor chiral superfield W_α and $\bar{W}^{\dot\alpha}$ to obtain that

$$W_\alpha(y) = \lambda_\alpha(y) + \theta_\alpha D(y) - i\,(\sigma^{\mu\nu}\theta)_\alpha\,F_{\mu\nu}(y) + i\,\theta^2\,(\sigma^\mu\,D_\mu\bar\lambda(y))_\alpha\,,$$
$$\bar{W}^{\dot\alpha}(y^\dagger) = \bar\lambda^{\dot\alpha}(y^\dagger) + \bar\theta^{\dot\alpha}D(y^\dagger) + i\,(\bar\sigma^{\mu\nu}\bar\theta)^{\dot\alpha}\,F_{\mu\nu}(y^\dagger) + i\,\bar\theta^2\,(\bar\sigma^\mu\,D_\mu\lambda(y^\dagger))^{\dot\alpha}\,,$$

where

$$D_\mu\lambda = \partial_\mu\lambda + i\,g\,[A_\mu\,,\,\lambda]\,, \quad D_\mu\bar\lambda = \partial_\mu\bar\lambda + i\,g\,[A_\mu\,,\,\bar\lambda]\,,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,g\,[A_\mu\,,\,A_\nu]\,.$$

Supersymmetric gauge kinetic action:

The chiral and anti-chiral nature of the Field strengths: W_α and $\bar{W}_{\dot\alpha}$, their spinor character and renormalizability leads to the following gauge-invariant action:

$$S_{\rm gauge} = \int d^4x \, \frac{1}{2} \left[{\rm Tr} \, (W^\alpha W_\alpha)|_{\theta^2} \, + \, {\rm Tr} \, (\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha})|_{\bar\theta^2} \right]$$
$$= \int d^4x \, d^4\theta \, \frac{1}{2} \left[{\rm Tr} \, (W^\alpha W_\alpha) \, \delta(\bar\theta) \, + \, {\rm Tr} \, (\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}) \, \delta(\theta) \right] \, .$$

For the U(1) case, replace: $\frac{1}{2} {\rm Tr} \, (W^\alpha W_\alpha) \to \frac{1}{4} W^\alpha W_\alpha$.

...

Exercises:

(i) Calculate $S_{\rm gauge}$ to find that

$$S_{\rm gauge} \; = \; -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \, + \, \bar\lambda^a i \bar\sigma^\mu D^{ab}_\mu \lambda^b \, + \, \frac{1}{2} D^a D^a \, ,$$

where
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$
is the SU(N) strength field tensor, and

$$D^{ab}_\mu \; = \; \partial_\mu \delta^{ab} \, + \, g f^{abc} A^c_\mu$$

is the covariant derivative in the adjoint rep. of SU(N). In the U(1) case, $f^{abc} = 0$ and $a, b = 1$.

(ii) Show that $\mathcal{L}_{\rm gauge}$ is invariant under the gauge trans:

$$\delta_\Lambda A^a_\mu \; = \; \partial_\mu \Lambda^a \, + \, g f^{abc} A^c_\mu \Lambda^b \, ,$$
$$\delta_\Lambda \lambda^a \; = \; g f^{abc} \lambda^b \Lambda^c \, , \qquad \delta_\Lambda D^a = g f^{abc} D^b \Lambda^c \, ,$$

where $\Lambda^a(x)$ are infinitesimal gauge parameters.

(iii) Show that $\mathcal{L}_{\rm gauge}$ is invariant under the SUSY trans:

$$\delta_\zeta A^a_\mu \; = \; -\bar\zeta \bar\sigma_\mu \lambda^a \, - \, \bar\lambda^a \bar\sigma_\mu \zeta \, ,$$
$$\delta_\zeta \lambda^a_\alpha \; = \; -i \, (\sigma^{\mu\nu} \zeta)_\alpha \, F^a_{\mu\nu} \, + \, \zeta_\alpha D^a \, ,$$
$$\delta_\zeta \bar\lambda^a_{\dot\alpha} \; = \; -i \, (\bar\zeta \sigma^{\mu\nu})_{\dot\alpha} \, F^a_{\mu\nu} \, + \, \bar\zeta_{\dot\alpha} D^a \, ,$$
$$\delta_\zeta D^a \; = \; -i \, \left(\bar\zeta \bar\sigma^\mu D^{ab}_\mu \lambda^b \, - \, D^{ab}_\mu \bar\lambda^b \bar\sigma^\mu \zeta \right) \, ,$$

where ζ_α and $\bar\zeta_{\dot\alpha}$ are infinitesimal x-independent Grassmann parameters.

– Gauge Interactions to Matter in SGTs

SUSY gauge trans of chiral and antichiral superfields:

$$\Phi(y,\theta) \;\rightarrow\; \Phi'(y,\theta) \;=\; e^{-2ig\Lambda(y,\theta)}\,\Phi(y,\theta)\,,$$
$$\Phi^\dagger(y^\dagger,\bar\theta) \;\rightarrow\; \Phi'^\dagger(y^\dagger,\bar\theta) \;=\; \Phi^\dagger(y^\dagger,\bar\theta)\,e^{2ig\Lambda^\dagger(y^\dagger,\bar\theta)}\,,$$

with $\bar D_{\dot\alpha}\Lambda(y,\theta) \;=\; 0$.

Hence, the Φ-kinetic term is not gauge invariant:

$$\mathrm{Tr}\,(\Phi^\dagger\Phi) \;\rightarrow\; \mathrm{Tr}\,(\Phi^\dagger e^{2ig\Lambda^\dagger}\,e^{-2ig\Lambda}\,\Phi) \;\neq\; \mathrm{Tr}\,(\Phi^\dagger\Phi)\,,$$

where the trace is taken over the group space in the fundamental rep of Φ: $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_N)$ for SU(N).

Including the SUSY gauge connection, e^{2gV}, we can write the following gauge invariant term:

$$\mathrm{Tr}\,(\Phi^\dagger\,e^{2gV}\,\Phi) \;\rightarrow\; \mathrm{Tr}\,(\Phi^\dagger e^{2ig\Lambda^\dagger}\,e^{-2ig\Lambda^\dagger}\,e^{2gV}\,e^{2ig\Lambda}\,e^{-2ig\Lambda}\,\Phi) \;=\; \mathrm{Tr}\,(\Phi^\dagger\,e^{2gV}\,\Phi)\,.$$

The corresponding SUSY gauge invariant action is

$$S^\Phi_{\rm kin} = \int d^4x\,\mathrm{Tr}\,(\Phi^\dagger\,e^{2gV}\,\Phi)|_{\theta^2\bar\theta^2} = \int d^4x\,d^4\theta\,\mathrm{Tr}\,(\Phi^\dagger\,e^{2gV}\,\Phi)\,.$$

The Φ-kinetic term in the WZ gauge:

Straightforward calculation of $\mathcal{L}^\Phi_{\rm kin} \;=\; \mathrm{Tr}\,(\Phi^\dagger\,e^{2gV}\,\Phi)|_{\theta^2\bar\theta^2}$ in the WZ gauge leads to a result that is equivalent to performing the following two additions in the ungauged WZ Lagrangian $\mathcal{L}_{\rm kin}$ presented on page 40.

The two additions:

(i) Couple $\widehat V^a \supset (A^a_\mu,\ \lambda^a,\ D^a)$ to $\widehat\Phi \supset (\phi,\ \xi,\ F)$ in a gauge- and SUSY- invariant way:

$$\mathcal{L}^{\widehat V\widehat\Phi}_{\rm int} = g(\phi^\dagger T^a\phi)\,D^a \;-\; \sqrt{2}\,g\,[\,(\phi^\dagger T^a\xi)\,\lambda^a \;+\; \bar\lambda^a(\bar\xi T^a\phi)\,]\,.$$

(ii) Change ordinary derivatives ∂_μ to covariant derivatives:

$$\partial_\mu \;\rightarrow\; D_\mu \equiv \partial_\mu \;+\; igA^a_\mu T^a\,,$$

in the Lagrangian with the scalar multiplets, i.e. make the substitutions in the WZ Lagrangian:

$$\partial_\mu\xi \;\rightarrow\; D_\mu\xi \;=\; \partial_\mu\xi \;+\; igA^a_\mu T^a\xi$$
$$\partial_\mu\phi \;\rightarrow\; D_\mu\phi \;=\; \partial_\mu\phi \;+\; igA^a_\mu T^a\phi$$
$$\partial_\mu\phi^\dagger \;\rightarrow\; (D_\mu\phi)^\dagger \;=\; \partial_\mu\phi^\dagger \;-\; igA^a_\mu\phi^\dagger T^a$$

The general SGT Lagrangian with matter fields:

$$S_{\rm SGT} \;=\; \int d^4x\, d^4\theta \,[\,\frac{1}{2}\,{\rm Tr}\,(W^\alpha W_\alpha)\,\delta(\bar\theta) \;+\; \frac{1}{2}\,{\rm Tr}\,(\bar W_{\dot\alpha} \bar W^{\dot\alpha})\,\delta(\theta) \;+\; {\rm Tr}\,(\Phi^\dagger\, e^{2gV}\,\Phi) \;+\; W(\Phi)\,\delta(\bar\theta) \;+\; W^\dagger(\Phi^\dagger)\,\delta(\theta)\,]\,,$$

where $W(\Phi)$ is the superpotential allowed by the gauge symmetries of the theory.

In summary, after integrating over the θ-space, we get

$$\begin{aligned}\mathcal{L}_{\rm SGT} \;=\;& -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \;+\; \bar\lambda^a i\bar\sigma^\mu D^{ab}_\mu \lambda^b \;+\; \frac{1}{2} D^a D^a \\ &+\; (D^\mu\phi)^\dagger (D_\mu\phi) \;+\; \bar\xi\, i\bar\sigma^\mu (D_\mu\xi) \;+\; F^\dagger F \\ &+\; g(\phi^\dagger T^a\phi)\, D^a \;-\; \sqrt{2}\, g\,[\,(\phi^\dagger T^a\xi)\,\lambda^a \;+\; \bar\lambda^a(\bar\xi T^a\phi)\,] \\ &-\; \frac{1}{2} W_{\phi\phi}\,\xi\xi \;+\; W_\phi\, F \;-\; \frac{1}{2} W^\dagger_{\phi\phi}\bar\xi\bar\xi \;+\; F^\dagger W^\dagger_\phi\,.\end{aligned}$$

The auxiliary fields F and D^a in $\mathcal{L}_{\rm SGT}$ can be eliminated by using the equations of motion:

$$F \;=\; -W^\dagger_\phi\,, \qquad D^a \;=\; -g(\phi^\dagger T^a \phi)\,.$$

The complete real potential V of a SGT then becomes

$$V \;=\; F^\dagger F + \frac{1}{2} D^a D^a \;=\; W_\phi W^\dagger_\phi \;+\; \frac{1}{2} g^2 (\phi^\dagger T^a\phi)^2 \;\ge\; 0$$

Note that the potential V is determined by other interactions in the theory! (More discussion on V in Chapter 6).

Exercises:

(i) Show that $\mathcal{L}_{\rm SGT}$ is invariant under the gauge covariant SUSY trans:

$$\begin{aligned}\delta_\zeta A^a_\mu \;&=\; -\,\bar\zeta\bar\sigma_\mu\lambda^a \;-\; \bar\lambda^a\bar\sigma_\mu\zeta\,, \\ \delta_\zeta \lambda^a_\alpha \;&=\; -\,i\,(\sigma^{\mu\nu}\zeta)_\alpha F^a_{\mu\nu} \;+\; \zeta_\alpha D^a\,, \\ \delta_\zeta \bar\lambda^a_{\dot\alpha} \;&=\; -\,i\,(\bar\zeta\sigma^{\mu\nu})_{\dot\alpha} F^a_{\mu\nu} \;+\; \bar\zeta_{\dot\alpha} D^a\,, \\ \delta_\zeta D^a \;&=\; -\,i\,\left(\bar\zeta\bar\sigma^\mu D^{ab}_\mu\lambda^b \;-\; D^{ab}_\mu\bar\lambda^b\bar\sigma^\mu\zeta\right)\,, \\ \delta_\zeta\phi \;&=\; \sqrt{2}\,\zeta\xi\,, \\ \delta_\zeta\xi_\alpha \;&=\; -i(\sigma^\mu\bar\zeta)_\alpha D_\mu\phi \;+\; \sqrt{2}\,\zeta_\alpha F\,, \\ \delta_\zeta\bar\xi_{\dot\alpha} \;&=\; i(\zeta\sigma^\mu)_{\dot\alpha}\,(D_\mu\phi)^\dagger \;+\; \sqrt{2}\,\bar\zeta_{\dot\alpha} F^\dagger\,, \\ \delta_\zeta F \;&=\; -i\,\sqrt{2}\,\bar\zeta\bar\sigma^\mu(D_\mu\xi) \;+\; 2g\,(T^a\phi)\bar\zeta\bar\lambda^a\,.\end{aligned}$$

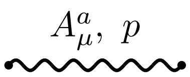
(ii) Show that the above SUSY algebra closes off-shell for gauge covariant objects:

$$[\,\delta_{\zeta_1}\,,\,\delta_{\zeta_2}\,]\,X \;=\; -2i\,(\zeta_1\sigma^\mu\bar\zeta_2 \;-\; \zeta_2\sigma^\mu\bar\zeta_1)\;D_\mu\cdot X\;,$$

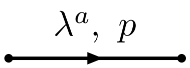
where $X = \phi, \xi, F, F^a_{\mu\nu}, \lambda^a, D^a$ and arbitrary covariant derivatives of X. The action of D_μ on X, $D_\mu\cdot X$, depends on the rep of X.

– Feynman rules

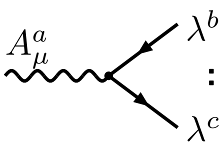
All Feynman rules for SGT can directly be read off from $\mathcal{L}_{\rm SGT}$.



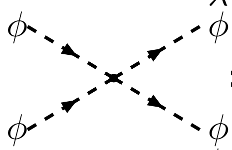
: $\dfrac{i\delta^{ab}}{p^2 + i\varepsilon}\left[-\eta_{\mu\nu} + (1-\xi)\,\dfrac{p_\mu p_\nu}{p^2}\right]$



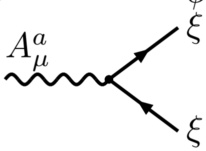
: $\dfrac{i\delta^{ab}}{\not p + i\varepsilon}$



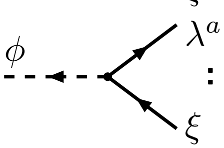
: $-g f^{abc}\gamma_\mu$



: $ig^2\, T^a_{ij}\, T^a_{kl} \;+\; \cdots$



: $ig\, T^a\, \gamma_\mu P_L$



: $-i\sqrt{2}\, g\, T^a$

6. Spontaneous Breaking Mechanisms of SUSY

– Spontaneous SUSY Breaking

If the vacuum $|0\rangle$ respects SUSY, it fullfils the conditions:

$$Q_\alpha|0\rangle \;=\; \bar{Q}_{\dot\alpha}|0\rangle \;=\; 0\,.$$

In global SUSY, the Hamiltonian is related to the generators Q_α and $\bar{Q}_{\dot\alpha}$:

$$H \;=\; P^0 \;=\; \frac{1}{4}\left(\bar{Q}_{\dot 1}Q_1 \;+\; Q_1\bar{Q}_{\dot 1} \;+\; \bar{Q}_{\dot 2}Q_2 \;+\; Q_2\bar{Q}_{\dot 2}\right)\,.$$

The vacuum energy can be formally computed as

$$\langle 0|H|0\rangle \;=\; \frac{1}{4}\Big(||Q_1|0\rangle||^2 + ||\bar{Q}_{\dot 1}|0\rangle||^2 + ||Q_2|0\rangle||^2 + ||\bar{Q}_{\dot 2}|0\rangle||^2\Big)\,.$$

From this, we deduce that $\langle 0|H|0\rangle \;\geq\; 0$.

If SUSY is *exact*, then $\langle 0|H|0\rangle \;=\; 0$, implying a vanishing cosmological constant!

But, nature is not fully supersymmetric, and SUSY must be broken either spontaneously or explicitly.

$\therefore$ If $\langle 0|H|0\rangle \;>\; 0$, then SUSY breaks *spontaneously*.

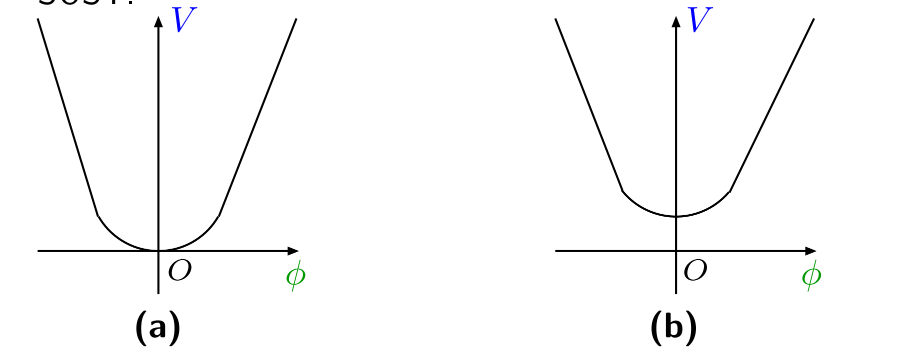
Exercise: Starting from the anti-commutation relation (v) on page 30, prove the above relation between H and the SUSY generators Q_α and $\bar{Q}_{\dot\alpha}$.

– O'Raifeartaigh Models

In Section 4, we have seen that the real scalar potential is

$$V \;=\; F_i^\dagger F_i \;+\; \frac{1}{2} D^a D^a \;=\; W_{\phi_i} W^\dagger_{\phi_i} \;+\; \frac{1}{2} g^2 (\phi_i^\dagger T^a \phi_i)^2 \;\geq\; 0\,.$$

Since $\langle 0|H|0\rangle = \langle 0|V|0\rangle$ in the absence of non-perturbative effects, then $\langle 0|V|0\rangle > 0$ will signify spontaneous breaking of SUSY.



The vacuum preserves SUSY — The vacuum breaks SUSY spontaneously

If there is no solution $F_i = 0$ and $D^a = 0$ for any values of the scalar fields ϕ_i, then SUSY is broken spontaneously.

Models that break SUSY through $F_i \neq 0$ are called O'Raifeartaigh models.

D-breaking of SUSY can be achieved only if the theory contains U(1) factors by means of the Fayet-Iliopoulos term.

Minimal O'Raifeartaigh Model

Such a model requires at least 3 chiral superfields, Φ_1, Φ_2 and Φ_3, and a superpotential of the form:

$$W(\Phi_1, \Phi_2, \Phi_3) \;=\; -\,\mu^2\,\Phi_1 \;+\; m\,\Phi_2\Phi_3 \;+\; \frac{1}{2}\lambda\,\Phi_1\Phi_3^2\,.$$

This model has a new type of $\mathrm{U}(1)_R$ symmetry:

$$\Phi_1 \to e^{i\varphi}\,\Phi_1\,, \quad \Phi_2 \to e^{i\varphi}\,\Phi_2\,, \quad \Phi_3 \to \Phi_3\,; \quad W \to e^{i\varphi} W\,.$$

The overall phase of W can be absorbed into $\delta(\bar\theta)$ that always multiplies W in the action (see page 53). This can be achieved by redefining $\theta_\alpha \to e^{i\frac{\varphi}{2}}\,\theta_\alpha$ and $\bar\theta_{\dot\alpha} \to e^{-i\frac{\varphi}{2}}\,\bar\theta_{\dot\alpha}$.

A symmetry under which the superpotential remains invariant up to an overall phase is called R-symmetry.

Furthermore, the model has the following discrete symmetry:

$$\Phi_1 \;\to\; \Phi_1\,, \qquad \Phi_2 \;\to\; -\,\Phi_2\,, \qquad \Phi_3 \;\to\; -\,\Phi_3\,.$$

The real scalar potential V of the model is given by

$$V \;=\; |F_1|^2 \;+\; |F_2|^2 \;+\; |F_3|^2\,,$$

where

$$F_1 = \mu^2 - \frac{1}{2}\lambda\,\phi_3^{*2}\,, \quad F_2 = -\,m\,\phi_3^*\,, \quad F_3 = -\,m\,\phi_2^* \;-\; \lambda\,\phi_1^*\,\phi_3^*\,.$$

Problem:

Consider the minimal O'Raifeartaigh model with 3 chiral superfields mentioned above.

(i) Verify the expressions for the auxiliary fields $F_{1,2,3}$ and show that all the conditions $F_{1,2,3} = 0$ cannot be simultaneously met.

(ii) Assume that $m^2 > \lambda\mu^2$, and show that the absolute minimum of the potential is at $\phi_2 = \phi_3 = 0$, with ϕ_1 being undetermined, i.e. ϕ_1 constitutes a 'flat direction' in the scalar potential. Is the R-symmetry broken in this case?

(iii) Find that for $m^2 > \lambda\mu^2$, the model predicts at the tree level 6 real scalars with squared masses:
$0\,, \quad 0\,, \quad m^2\,, \quad m^2 + \lambda\mu^2\,, \quad m^2 - \lambda\mu^2\,,$
and 3 Weyl fermions with masses: $0\,, \quad m\,, \quad m.$

[Hint: Consider appropriately the ungauged version of the Lagrangian $\mathcal{L}_{\rm SGT}$ given on page 53.]

(iv) What is the physical meaning of the massless Weyl fermion?

(v)* Calculate the mass spectrum of the model for the case $m^2 < \lambda\mu^2$. Does the ground state preserve the R-symmetry?

– The Fayet–Iliopoulos Term

The $\theta^2\bar{\theta}^2$-compenent $D'(x)$ of a U(1) vector superfield $V'(x,\theta,\bar{\theta})$ is both invariant under SUSY and gauge trans. (*Question*: Why?)

This allows us to add to the action a term linear in V', the so-called Fayet–Iliopoulos (FI) term:

$$S_{\rm FI} \;=\; \int d^4x\; \kappa V'|_{\theta^2\bar{\theta}^2} \;=\; \int d^4x\, d^4\theta\; \kappa V'\,.$$

The resulting scalar potential is

$$V_{\rm FI} \;=\; -\frac{1}{2}D'^2 \;-\; \kappa\, D' \;-\; D' g' \sum_i y_i\, \phi_i^*\phi_i\;,$$

where y_i are the U(1) charges of scalar fields. For instance, for the SM, these will be the U(1)$_Y$ hypercharges.

Using the equation of motions for D', we find that

$$D' \;=\; -\,\kappa \;-\; \sum_i y_i\, \phi_i^*\phi_i\;,$$

and hence

$$V_{\rm FI} \;=\; \frac{1}{2}\, D'^2 \;=\; \frac{1}{2}\left(\kappa \;+\; \sum_i y_i\, \phi_i^*\phi_i\right)^2\,.$$

Remark: The FI term does not break by itself SUSY, since one can have $D' = 0$, for specific values of the scalar fields. However, the synergy of D' with other F terms of the theory can break SUSY spontaneously.

...

Problem:

Consider a SQED model with two chiral superfields of opposite charge that includes a FI term:

$$\mathcal{L}_{\rm SQED} = (\,\Phi_1^\dagger\, e^{2eV}\,\Phi_1 \,+\, \Phi_2^\dagger\, e^{-2eV}\,\Phi_2 \,+\, \kappa\, V\,)|_{\theta^2\bar\theta^2} + m\,(\,\Phi_1\Phi_2|_{\theta^2} + \Phi_1^\dagger\Phi_2^\dagger|_{\bar\theta^2}\,)\,.$$

(i) Calculate the total real potential V and show that SUSY has to be broken spontaneously.

(ii) Consider the two cases: (a) $m^2 > e\kappa$ and (b) $m^2 < e\kappa$, and show that only SUSY is broken in (a), whereas in case (b) both SUSY and gauge symmetry are broken spontaneously.

(ii) Calculate the bosonic and fermionic mass spectrum of the model for the above two cases (a) and (b).

[Hint: For a discussion of the above problem, see the textbook by Wess and Bagger, pages 52–56.]

7. Minimal Supersymmetric Standard Model

– Model–Building of the MSSM

The MSSM is based on the SU(3)$_c\otimes$SU(2)$_L\otimes$U(1)$_Y$ gauge group, with the following field content:

Superfields	Bosons	Fermions	SU(3)$_c\otimes$SU(2)$_L\otimes$U(1)$_Y$
GAUGE			
$\widehat{G}$	$G^a_\mu\,\frac{1}{2}\lambda^a$	$\tilde{g}^a$	$(8,1,0)$
$\widehat{W}$	$W^i_\mu\,\frac{1}{2}\sigma_i$	$\widetilde{W}^i$	$(1,3,0)$
$\widehat{B}$	B_μ	$\widetilde{B}$	$(1,1,0)$
MATTER			
$\widehat{L}$	$\widetilde{L}^T = (\tilde\nu_l,\tilde l)_L$	$L^T = (\nu_l, l)_L$	$(1,2,-1)$
$\widehat{E}$	$\widetilde{E} = \tilde l^*_R$	$(e_R)^C = (e^C)_L$	$(1,1,2)$
$\widehat{Q}$	$\widetilde{Q}^T = (\tilde u,\tilde d)_L$	$Q^T = (u,d)_L$	$(3,2,\frac{1}{3})$
$\widehat{U}$	$\widetilde{U} = \tilde u^*_R$	$(u_R)^C = (u^C)_L$	$(3,1,-\frac{4}{3})$
$\widehat{D}$	$\widetilde{D} = \tilde d^*_R$	$(d_R)^C = (d^C)_L$	$(3,1,\frac{2}{3})$
$\widehat{H}_1$	$\widetilde{\Phi}_1^T = (\phi_1^{0*}, -\phi_1^-)$	$(\bar\psi^0_{H_1}, \psi^-_{H_1})$	$(1,2,-1)$
$\widehat{H}_2$	$\Phi_2^T = (\phi_2^+, \phi_2^0)$	$(\psi^+_{H_2}, \psi^0_{H_2})$	$(1,2,1)$

Note that from now on superfields will be denoted by a carret for notational convenience.

Remark: The generation of up- and down-quark masses from an holomorphic superpotential and the cancellation of anomalies due to the presence of higgsinos require that at least *two* Higgs doublets with opposite hypercharge be added to the theory.

The construction of the MSSM Lagrangian

$$S_{\rm MSSM} = \int d^4x\, d^4\theta\, \{\, [\, \frac{1}{2}\,{\rm Tr}(G^\alpha G_\alpha)\,\delta(\bar\theta) + \frac{1}{2}\,{\rm Tr}(W^\alpha W_\alpha)\,\delta(\bar\theta) + \frac{1}{4}\,B^\alpha B_\alpha\,\delta(\bar\theta)\ +\ {\rm h.c.}\,] + [\,\widehat{Q}^\dagger\,(e^{2g_s\widehat{G}} + e^{2g_w\widehat{W}} + e^{y_{Q_i}\,g'\widehat{B}})\,\widehat{Q} + \sum_{i=1,2}\widehat{H}_i^\dagger\,(e^{2g_w\widehat{W}} + e^{y_{H_i}\,g'\widehat{B}})\,\widehat{H}_i\ +\ \cdots\,] + W\,\delta(\bar\theta)\ +\ W^\dagger\,\delta(\theta)\ +\ \delta(\theta)\,\delta(\bar\theta)\,\mathcal{L}_{\rm soft}\ \}$$

where G_α, W_α and B_α are the SUSY SU(3)$_c$, SU(2)$_L$ and U(1)$_Y$ field strengths, respectively.

In addition, W is the MSSM superpotential

$$W = h_l\,\widehat{H}_1^T i\sigma_2\widehat{L}\widehat{E} + h_d\,\widehat{H}_1^T i\sigma_2\widehat{Q}\widehat{D} + h_u\,\widehat{Q}^T i\sigma_2\widehat{H}_2\widehat{U} - \mu\,\widehat{H}_1^T i\sigma_2\widehat{H}_2\,,$$

and $\mathcal{L}_{\rm soft}$ is the soft SUSY-breaking Lagrangian

$$-\mathcal{L}_{\rm soft} = \frac{1}{2}\,(m_{\tilde g}\,\lambda^a_{\tilde g}\lambda^a_{\tilde g}\ +\ m_{\widetilde W}\,\lambda^i_{\widetilde W}\lambda^i_{\widetilde W}\ +\ m_{\widetilde B}\,\lambda_{\widetilde B}\lambda_{\widetilde B}\ +\ {\rm h.c.}) + \widetilde{M}^2_L\,\widetilde{L}^\dagger\widetilde{L}\ +\ \widetilde{M}^2_Q\,\widetilde{Q}^\dagger\widetilde{Q}\ +\ \widetilde{M}^2_U\,\widetilde{U}^*\widetilde{U}\ +\ \widetilde{M}^2_D\,\widetilde{D}^*\widetilde{D} + \widetilde{M}^2_E\,\widetilde{E}^*\widetilde{E}\ +\ m_1^2\,\widetilde{\Phi}_1^\dagger\widetilde{\Phi}_1\ +\ m_2^2\,\Phi_2^\dagger\Phi_2 - (\,B\mu\,\widetilde{\Phi}_1^T i\sigma_2\Phi_2\ +\ {\rm h.c.}\,)\ +\ (\,h_lA_l\,\Phi_1^\dagger\widetilde{L}\widetilde{E} + h_dA_d\,\Phi_1^\dagger\widetilde{Q}\widetilde{D}\ -\ h_uA_u\,\Phi_2^T i\sigma_2\widetilde{Q}\widetilde{U}\ +\ {\rm h.c.}\,)\,,$$

with $\widetilde{\Phi}_1 = i\sigma_2\Phi_1^*$ and $\Phi_1^T = (\phi_1^\dagger, \phi_1^0)$.

Global Symmetries of the MSSM

The MSSM has two softly broken global symmetries:

(i) The Peccei–Quinn (PQ) symmetry $U(1)_{\rm PQ}$:

$$\widehat{H}_1\,(1),\ \widehat{H}_2\,(1),\ \widehat{Q}\,(-1),\ \widehat{U}\,(0),\ \widehat{D}\,(0),\ \widehat{L}\,(-1),\ \widehat{E}\,(0)\,.$$

The PQ symmetry is broken by the μ and $B\mu$ parameters.

(ii) The R-symmetry $U(1)_R$:

$$\widehat{H}_1\,(0),\ \widehat{H}_2\,(0),\ \widehat{Q}\,(1),\ \widehat{U}\,(1),\ \widehat{D}\,(1),\ \widehat{L}\,(1),\ \widehat{E}\,(1)\,,$$

implying that $W_{\rm MSSM}\,(2)$. The R-symmetry is broken by the μ parameter, the *trilinear* soft SUSY-breaking terms and the gaugino masses. (*Question*: Why?)

In addition, the MSSM possesses an *exact* discrete $\mathbb{Z}_2$-matter parity, better known as R-parity. Under $\mathbb{Z}_2$, all ordinary SM particles have charge $+1$, while all SUSY partners have charge -1.

(*Question*: What are the phenomenological consequences of R-parity conservation?)

Exercise: Show that the additional SUSY operators:

$$W_{\not R} = \varepsilon_i\widehat{L}_i^T i\sigma_2\widehat{H}_2 + \lambda_{ijk}\widehat{L}_i^T i\sigma_2\widehat{L}_j\widehat{E}_k + \lambda'_{ijk}\widehat{L}_i^T i\sigma_2\widehat{L}_j\widehat{D}_k + \lambda''_{ijk}\widehat{U}_i\widehat{U}_j\widehat{D}_k$$

break R parity. Which operators break the lepton and baryon numbers idividually?

– Gauge-Coupling Unification

The one-loop Renormalization–Group (RG) equations for the SM gauge couplings are:

$$\frac{dg_{1,2,3}}{dt} = \frac{b_{1,2,3}}{16\pi^2}\, g^3_{1,2,3} \quad \Rightarrow \quad \frac{d\alpha^{-1}_{1,2,3}}{dt} = -\frac{b_{1,2,3}}{2\pi}\,,$$

where $t = \ln(Q/M_Z)$ and $\alpha_{1,2,3} = \frac{g^2_{1,2,3}}{4\pi}$, with $g_1 = \sqrt{\frac{5}{3}}\, g'$, $g_2 = g_w$ and $g_3 = g_s$.

The normalization of g_1 is chosen so as to agree with the covariant derivative when embedding the SM into an SU(5) or SO(10) unified theory.

The $b_{1,2,3}$ constants are:

$$b^{\rm SM}_{1,2,3} = \left(\frac{41}{10}\,,\ -\frac{19}{6}\,,\ -7\right)\,, \quad b^{\rm MSSM}_{1,2,3} = \left(\frac{33}{5}\,,\ 1\,,\ -3\right)\,,$$

Problem (*Gauge-coupling unification*):

Given that $\alpha_s(M_Z) = 0.12$, $\alpha_w(M_Z) = 0.033$ and $\alpha_{\rm em}(M_Z) = 1/128$, calculate:

(i) the intersection point $M_{\rm U}$ due to RG evolution of the g_2 and g_3 couplings in the SM and the MSSM.

(ii) the coupling $g_1(M_Z)$, assuming that $g_1(M_{\rm U}) = g_2(M_{\rm U}) = g_3(M_{\rm U})$. Compare then your prediction for $\alpha_{\rm em}(M_Z)$ in the SM and the MSSM, with its experimental value.

– The MSSM Higgs Potential

I. *Tree-Level Potential*

The MSSM Higgs potential can be computed by

$$V_{\rm MSSM} = W_{\widetilde{\Phi}_1} W^\dagger_{\widetilde{\Phi}_1} + W_{\Phi_2} W^\dagger_{\Phi_2} + \frac{1}{2}\left[\sum_{i=1}^{3} (D^i)^2 + D'^2\right] - \mathcal{L}^{\rm Higgs}_{\rm soft}\,,$$

where

$$\begin{aligned}
W_{\widetilde{\Phi}_1} &= \mu\, \Phi_2^T\, i\sigma_2 + \cdots\,,\\
W_{\Phi_2} &= -\mu\, \widetilde{\Phi}_1^T\, i\sigma_2 + \cdots\,,\\
D^i &= \frac{g_w}{2}\left(\widetilde{\Phi}_1^\dagger\, \sigma_i\, \widetilde{\Phi}_1 + \Phi_2^\dagger\, \sigma_i\, \Phi_2 + \cdots\right)\,,\\
D' &= \frac{g'}{2}\left(-\,\widetilde{\Phi}_1^\dagger\, \widetilde{\Phi}_1 + \Phi_2^\dagger\, \Phi_2 + \cdots\right)\,,\\
-\mathcal{L}^{\rm Higgs}_{\rm soft} &= m_1^2\, \widetilde{\Phi}_1^\dagger \widetilde{\Phi}_1 + m_2^2\, \Phi_2^\dagger \Phi_2 - \left(B\mu\, \widetilde{\Phi}_1^T i\sigma_2 \Phi_2 + {\rm h.c.}\right)\,,
\end{aligned}$$

and the dots stand for non-Higgs terms.

Notice that the quartic couplings of the Higgs potential are fully determined by the SU(2)$_L$ and U(1)$_Y$ gauge couplings g_w and g'.

Problem (The Tree-Level Higgs Potential):

(i) Use the identity

$$\sum_{i=1}^{3}(\sigma_i)_{ab}\,(\sigma_i)_{cd} \;=\; 2\,\delta_{ad}\,\delta_{cb} \;-\; \delta_{ab}\,\delta_{cd}$$

to cast the Largangian $\mathcal{L}_V^0$ containing the tree-level Higgs potential ($\mathcal{L}_V^0 = -V_{\rm MSSM}$) into the form:

$$\begin{aligned}\mathcal{L}_V^0 \;=\;& \mu_1^2(\Phi_1^\dagger\Phi_1) \,+\, \mu_2^2(\Phi_2^\dagger\Phi_2) \,+\, m_{12}^2(\Phi_1^\dagger\Phi_2) \,+\, m_{12}^{*2}(\Phi_2^\dagger\Phi_1)\\ &+\lambda_1(\Phi_1^\dagger\Phi_1)^2 \,+\, \lambda_2(\Phi_2^\dagger\Phi_2)^2 \,+\, \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2)\\ &+\lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\,,\end{aligned}$$

with

$$\mu_1^2 \;=\; -m_1^2 - |\mu|^2\,, \qquad \mu_2^2 \;=\; -m_2^2 - |\mu|^2\,, \qquad m_{12}^2 \;=\; B\mu\,,$$

$$\lambda_1 \;=\; \lambda_2 \;=\; -\frac{1}{8}\,(g_w^2 + g'^2)\,, \qquad \lambda_3 \;=\; -\frac{1}{4}\,(g_w^2 - g'^2)\,,$$

$$\lambda_4 \;=\; \frac{1}{2}\,g_w^2\,.$$

(ii) Use the linear expansions of the Higgs doublets about the ground state:

$$\Phi_1 \;=\; \begin{pmatrix} \phi_1^+ \\ \frac{1}{\sqrt{2}}\,(v_1 + \phi_1 + ia_1) \end{pmatrix}\,,$$

$$\Phi_2 \;=\; e^{i\xi}\begin{pmatrix} \phi_2^+ \\ \frac{1}{\sqrt{2}}\,(v_2 + \phi_2 + ia_2) \end{pmatrix}$$

to calculate the so-called tadpole parameters:

$$\begin{aligned}T_{\phi_{1(\phi_2)}} \;\equiv\; \left\langle \frac{\partial\mathcal{L}_V}{\partial\phi_{1(2)}}\right\rangle \;=\;& v_{1(2)}\Bigg[\mu_{1(2)}^2 \,+\, \frac{v_1 v_2}{v_{1(2)}^2}\,{\rm Re}(m_{12}^2 e^{i\xi})\\ &+\lambda_{1(2)}v_{1(2)}^2 \,+\, \frac{1}{2}\,(\lambda_3+\lambda_4)v_{2(1)}^2\Bigg]\,,\end{aligned}$$

$$T_{a_{1(a_2)}} \;\equiv\; \left\langle \frac{\partial\mathcal{L}_V}{\partial a_{1(2)}}\right\rangle \;=\; +(-)\,v_{2(1)}\,{\rm Im}(m_{12}^2 e^{i\xi})\,.$$

(iii) The tree-level MSSM mass spectrum consists of a degenerate pair of charged Higgs bosons $H^\pm$, two CP-even Higgs bosons h and H, and one CP-odd scalar A.

Require the vanishing of the tadpole parameters to obtain:

$$M_A^2 \;=\; \frac{{\rm Re}(m_{12}^2 e^{i\xi})}{s_\beta\,c_\beta}\,, \quad M_{H^\pm}^2 = M_A^2 + M_W^2$$

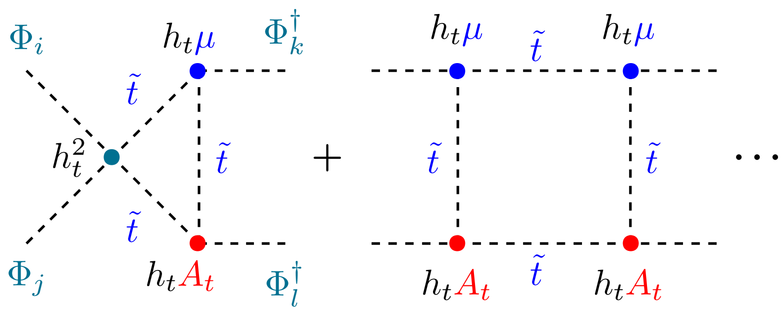
$$M_{h(H)}^2 \;=\; \frac{1}{2}\,\Big(M_A^2 + M_Z^2 \;-(+)\;\sqrt{(M_Z^2+M_A^2)^2 - 4M_Z^2M_A^2c_{2\beta}}\Big)\,,$$

where $\tan\beta = \frac{s_\beta}{c_\beta} = \frac{v_2}{v_1}$, $v = \sqrt{v_1^2+v_2^2} = 2M_W/g_w$ and $M_Z = \frac{1}{2}\sqrt{g_w^2+g'^2}\,v$.

(iv) Use the above results to show that $M_h \le M_Z$ and $M_h \le M_A$. From LEP2, we know that $M_h > 114$ GeV, so the MSSM is already ruled out at the tree level!
But, radiative effects are large and may rescue us!

II. *Quantum Corrections to the Higgs Potential*

Radiative effects on the MSSM Higgs potential are large:



$\subset \quad \mathcal{L}_V(\Phi_1, \Phi_2)$

The radiatively-corrected upper bound on M_h becomes:

$$M_h^2 \ \leq\ M_Z^2 c_{2\beta}^2 \left(1 - \frac{3h_t^2}{8\pi^2}\, t\right) + \frac{3h_t^4 v^2 s_\beta^4}{8\pi^2} \Big\{ \Big(1 + \frac{4\alpha_s}{3\pi}\frac{X_t}{M_S}\Big)\Big[\, t + \frac{X_t^2}{M_S^2}\Big(1 - \frac{X_t^2}{12M_S^2}\Big)\Big] + \frac{1}{16\pi^2}\Big(\tfrac{3}{2}h_t^2 - 32\pi\alpha_s\Big)\Big[\frac{2X_t^2}{M_S^2}\Big(1 - \frac{X_t^2}{12M_S^2}\Big)t + t^2\Big]\Big\}\,,$$

where $t = \ln(M_S^2/m_t^2)$, $X_t = A_t - \mu/t_\beta$ is the stop-mixing parameter and M_S is the soft SUSY-breaking scale.

For $M_S \approx 1$ TeV, $X_t \approx 2.45$ TeV and $m_t = \frac{1}{\sqrt{2}}h_t v_2 \approx 175$ GeV, we have

$$M_h \ \lesssim\ 115\ (135)\ \mathrm{GeV}\,,$$

for $\tan\beta = 3\ (20)$.

– Radiative Breaking of Gauge Symmetry

Radiative effects may also break the electroweak gauge symmetry by flipping the sign of the operator $\mu_2^2 \Phi_2^\dagger \Phi_2$ from positive to negative.

The leading effects of the radiative corrections can be calculated by means of RG equations.

For illustration, let us consider the dominant t-Yukawa RG effects on the soft SUSY-breaking mass $m_2^2 \approx \mu_2^2$ (with $|\mu|^2 \ll m_2^2$) and on the left-handed and right-handed soft stop masses $\widetilde{M}_{Q_3}^2 \equiv m_L^2$ and $\widetilde{M}_{U_3}^2 \equiv m_R^2$:

$$\frac{d}{dt}\begin{pmatrix} m_2^2 \\ m_R^2 \\ m_L^2 \end{pmatrix} = \frac{h_t^2}{8\pi^2}\begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} m_2^2 \\ m_R^2 \\ m_L^2 \end{pmatrix} + \frac{h_t^2 |A_t^2|}{8\pi^2}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

where $t = \ln(Q/M_U)$.

Exercise: Assume that $A_t = 0$ and a common mass scale $m_2^2 = m_R^2 = m_L^2 = m_0^2$ at $M_U \approx 10^{16}$ GeV to solve the above RG equation. Evaluate the solution at $Q = 1$ TeV to approximately find that

$$\begin{pmatrix} m_2^2 \\ m_R^2 \\ m_L^2 \end{pmatrix} \approx \frac{1}{2}m_0^2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

i.e. m_2^2 flips sign and so triggers ***radiative electroweak symmetry breaking***.

– Soft Radiative Breaking of CP Symmetry

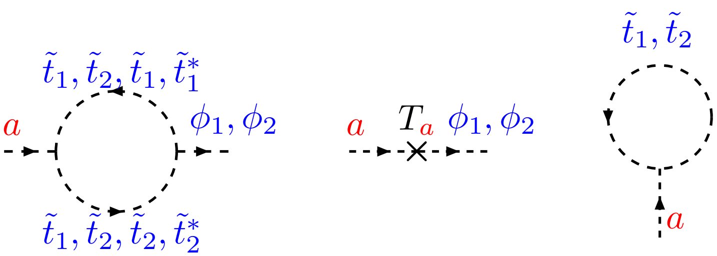
[A.P. '98; A.P., C. Wagner '99 . . .]

Radiative effects from CP-violating soft SUSY-breaking terms may break the CP symmetry of the tree-level Higgs potential.

There are two kinds of radiative CP-violating effects:
I. CP-violating self-energy effects,
II. CP-violating vertex effects.

. . .

I. *CP-violating self-energy effects*



$$\mathcal{M}^2_{SP} \sim \frac{m_t^4}{v^2} \frac{\mathrm{Im}\,(\mu A_t)}{32\pi^2 Q_t^2} \times \left(1,\ \frac{|A_t|^2}{Q_t^2},\ \frac{|\mu|^2}{\tan\beta\ Q_t^2},\ \frac{2\mathrm{Re}\,(\mu A_t)}{Q_t^2}\right) \lesssim (100\ \mathrm{GeV})^2$$

Exercise: Use the CP-odd tadpole minimization conditions $T_{a_{1,2}} = 0$ to show that the tree-level MSSM Higgs potential is CP-invariant.

The mixing of the three neutral Higgs bosons

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ a \end{pmatrix} = O \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$$

O is a 3×3 orthogonal matrix which also describes the mixing of the Higgs bosons with different CP parities.

In analogy to the case of neutrinos and quarks, Higgs bosons with mixed CP parities are ordered according to their weights:

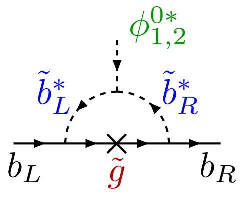
$$M_{H_1} \le M_{H_2} \le M_{H_3}$$

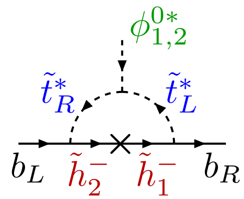
At the one-loop level, M_{H_i} (with $i = 1, 2, 3$) and O are analytically determined by the input parameters:

$$M_{H^+}(m_t)\,, \quad \tan\beta(m_t)\,,$$
$$\mu(Q_{tb})\,, \quad A_t(Q_{tb})\,, \quad A_b(Q_{tb})\,,$$
$$\widetilde{M}^2_Q(Q_{tb})\,, \quad \widetilde{M}^2_t(Q_{tb})\,, \quad \widetilde{M}^2_b(Q_{tb})\,.$$

II. *CP-violating vertex effects*

Effective $H_1 bb$-coupling:





$$-\mathcal{L}^{\rm eff}_{\phi^0 \bar{b}b} = (h_b + \delta h_b)\,\phi_1^{0*}\,\bar{b}_R b_L + \Delta h_b\,\phi_2^{0*}\,\bar{b}_R b_L \;+\; \text{h.c.}$$

with

$$\frac{\delta h_b}{h_b} \sim -\frac{2\alpha_s}{3\pi}\frac{m_{\tilde{g}}^* A_b}{\max(Q_b^2, |m_{\tilde{g}}|^2)} - \frac{|h_t|^2}{16\pi^2}\frac{|\mu|^2}{\max(Q_t^2, |\mu|^2)}$$

$$\frac{\Delta h_b}{h_b} \sim \frac{2\alpha_s}{3\pi}\frac{m_{\tilde{g}}^* \mu^*}{\max(Q_b^2, |m_{\tilde{g}}|^2)} + \frac{|h_t|^2}{16\pi^2}\frac{A_t^* \mu^*}{\max(Q_t^2, |\mu|^2)}$$

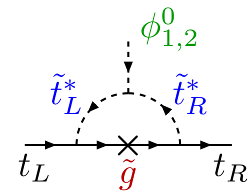
Exercise: Consider the condition:

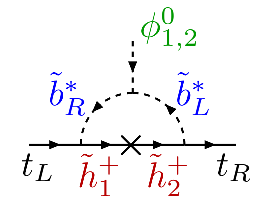
$$\mathcal{L}^{\rm eff}_{\phi^0 \bar{b}b}\,|_{\phi^0_{1,2} = \frac{1}{\sqrt{2}} v_{1,2}} \;=\; -\,m_b\,\bar{b}\,b$$

to obtain the effective b-quark Yukawa coupling:

$$h_b \;=\; \frac{g_w m_b}{\sqrt{2} M_W \cos\beta\,[\,1 + \delta h_b/h_b + (\Delta h_b/h_b)\tan\beta\,]}\,.$$

Effective $H_1 tt$-coupling:





$$-\mathcal{L}^{\rm eff}_{\phi^0 \bar{t}t} = \Delta h_t\,\phi_1^{0}\,\bar{t}_R t_L + (h_t + \delta h_t)\,\phi_2^{0}\,\bar{t}_R t_L \;+\; \text{h.c.}$$

with

$$\frac{\Delta h_t}{h_t} \sim \frac{2\alpha_s}{3\pi}\frac{m_{\tilde{g}}^* \mu^*}{\max(Q_t^2, |m_{\tilde{g}}|^2)} + \frac{|h_b|^2}{16\pi^2}\frac{A_b^* \mu^*}{\max(Q_b^2, |\mu|^2)}$$

$$\frac{\delta h_t}{h_t} \sim -\frac{2\alpha_s}{3\pi}\frac{m_{\tilde{g}}^* A_t}{\max(Q_t^2, |m_{\tilde{g}}|^2)} - \frac{|h_b|^2}{16\pi^2}\frac{|\mu|^2}{\max(Q_b^2, |\mu|^2)}$$

Exercise: As in the previous exercise, consider an analogous condition to derive the effective t-quark Yukawa coupling:

$$h_t \;=\; \frac{g_w m_t}{\sqrt{2} M_W \sin\beta\,[\,1 + \delta h_t/h_t + (\Delta h_t/h_t)\cot\beta\,]}\,.$$

– Phenomenological Implications:

- Contributions to Electric Dipole Moments
- FCNC observables:
  $\Delta M_{K,B}$, ϵ_K, ϵ'/ϵ, $\mathcal{B}(B_{d,s} \to \ell^+\ell^-)$,
  $\mathcal{A}_{\rm CP}(B_{d,s} \to \ell^+_{L(R)}\ell^-_{L(R)})$, with $\ell = \mu\,,\tau$,
  $\mathcal{B}(B \to X_s\gamma)$, . . .
- Higgs phenomenology at LEP2, Tevatron, LHC and e^+e^- Linear Colliders.

  CPsuperH: a Super–Code for Higgs Phenomenology in the MSSM with Explicit CP Violation [hep-ph/0307377]

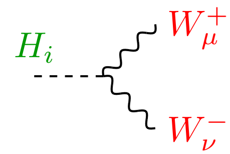
  J.S. Lee, A.P., M. Carena, S.Y. Choi, M. Drees, J. Ellis and C.E.M. Wagner

⋮

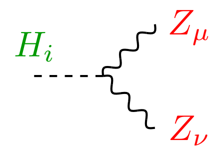
Higgs Phenomenology at High-Energy Colliders

Effective Higgs couplings to gauge bosons and fermions:

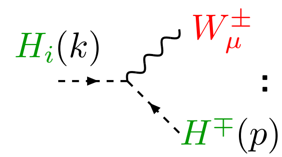
[M. Carena, J. Ellis, A.P., C. Wagner '00]



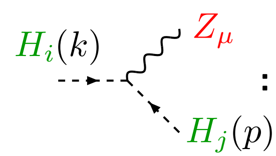
: $\quad i g_w M_W\,(c_\beta O_{1i} \,+\, s_\beta O_{2i})\, g_{\mu\nu}$



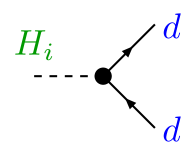
: $\quad i g_w\, \frac{M_W^2}{M_Z}\,(c_\beta O_{1i} \,+\, s_\beta O_{2i})\, g_{\mu\nu}$



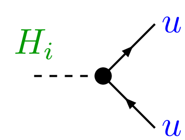
: $\quad \pm\frac{i}{2} g_w\,(c_\beta O_{2i} - s_\beta O_{1i} \,+\, i O_{3i})(p-k)_\mu$



: $$-\frac{i g_w M_Z}{2 M_W}\,[\, O_{3i}\,(c_\beta O_{2j} - s_\beta O_{1j}) \;-\; (i \leftrightarrow j)\,]\,(p-k)_\mu$$

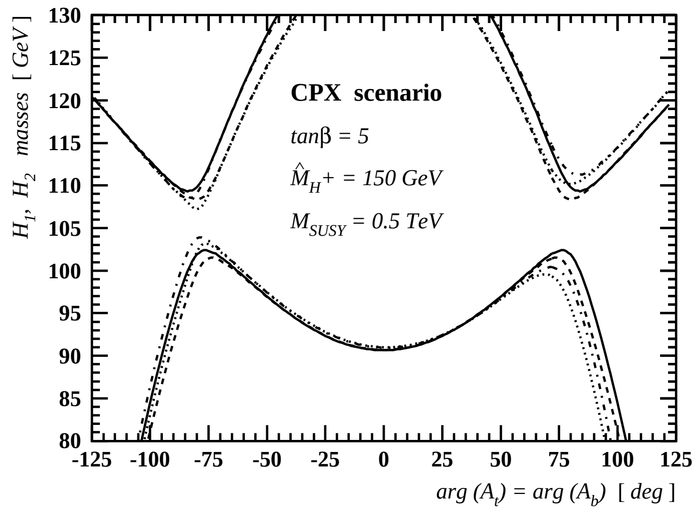


: $\quad -\frac{i g_w m_d}{2 M_W\, c_\beta}\,(O_{1i} \,-\, i s_\beta\, O_{3i}\,\gamma_5) \,+\, \cdots$

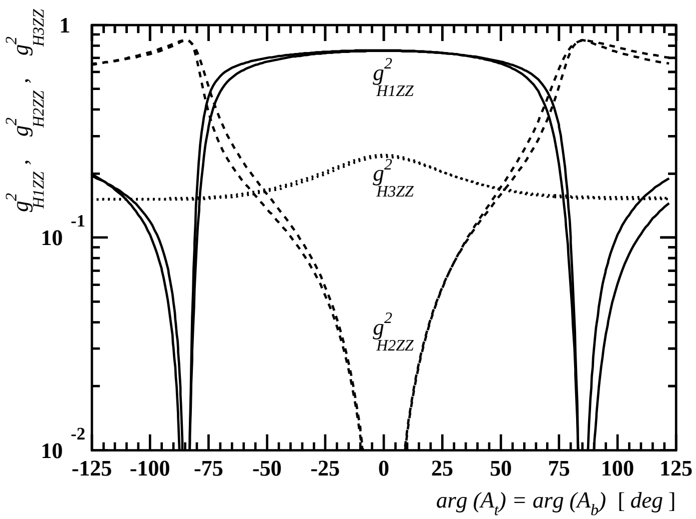


: $\quad -\frac{i g_w m_u}{2 M_W\, s_\beta}\,(O_{2i} \,-\, i c_\beta\, O_{3i}\,\gamma_5) \,+\, \cdots$

Generated with CPsuperH



(a)



(b)

8. SUperGRAvity

– Local Supersymmetry

So far, we have considered Lagrangians that are symmetric under a global trans of SUSY, namely the group parameter ζ was constant.

Local SUSY emerges by allowing the Grassman-valued group parameter ζ to become x-dependent. For example, neglecting gravity, the local SUSY trans have the usual form

$$\begin{aligned} \delta_\zeta \phi(x) &= \sqrt{2}\,\zeta(x)\,\xi(x) \\ \delta_\zeta \xi_\alpha(x) &= -i\,(\sigma^\mu\,\bar{\zeta}(x))_\alpha\, D_\mu\phi \,+\, \sqrt{2}\,\zeta_\alpha(x)\,F(x) \qquad \text{etc.} \end{aligned}$$

Hence, ζ has been promoted to a Weyl spinor, the Goldstino of spin 1/2.

The gravity multiplet associated with local SUSY contains the massless graviton $g_{\mu\nu}$ of spin 2 and the massless gravitino ψ_μ of spin 3/2 (ψ_μ is a 4-vector Weyl spinor, where the spinor index is suppressed).

The gravitino $\psi_\mu(x)$ acquires its mass by 'eating' the two dofs of Goldstino $\zeta(x)$. This is the famous **Super–Higgs Mechanism**.

A massive gravitino $\psi_\mu(x)$ has $2s + 1 = 2\,\frac{3}{2} + 1 = 4$ dofs: 2 dofs from the initially massless gravitino and 2 from the massless Goldstino.

– Non-renormalizable Interactions and Kähler Potential

The gravity sector of the simplest (on-shell) SUGRA model may be described by the Lagrangian:

$$\mathcal{L}_{\rm SUGRA} = -\frac{1}{2\kappa^2}\sqrt{-g}\,R - \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\,(\bar{\psi}_\mu\bar{\sigma}_\nu\mathcal{D}_\lambda\psi_\rho - \psi_\mu\sigma_\nu\bar{\mathcal{D}}_\lambda\bar{\psi}_\rho),$$

where $\kappa^2 = 8\pi G_N = 8\pi/M^2_{\rm Planck}$, $g = \det[g_{\mu\nu}(x)]$ and R is the Ricci curvuture scalar.

In addition, $\mathcal{D}_\lambda \equiv \partial_\lambda + \omega^{\alpha\beta}_\lambda\sigma_{\alpha\beta}$ is the covariant derivative w.r.t. local Lorentz trans and $\omega^{\alpha\beta}_\lambda$ is the corresponding affine connection or *spin connection*.

The local Minkowski metric $g_{\mu\nu}(x)$ can be written in terms of the *vielbeins* e^α_μ as $g_{\mu\nu} = e^\alpha_\mu\, e^\beta_\nu\eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the flat-space metric.

Without proof, we state that $\mathcal{L}_{\rm SUGRA}$ is invariant under the local SUSY trans (suppressing all spinor indices):

$$\begin{aligned}
\delta_\zeta e^\alpha_\mu &= -i\,\kappa\,(\,\bar{\zeta}\bar{\sigma}^\alpha\psi_\mu + \zeta\sigma^\alpha\bar{\psi}_\mu\,)\,,\\
\delta_\zeta\psi_\mu &= \frac{2}{\kappa}\mathcal{D}_\mu\,\zeta\,,\\
\delta_\zeta\omega^{\alpha\beta}_\lambda &= 0\,,
\end{aligned}$$

with $\zeta = \zeta(x)$.

Matter Sector of SUGRA:

Here, we summarize the main results:

(i) The superpotential $W(\Phi)$ is generalized to an arbitrary *non-renormalizable* hermitian function $G(\Phi_i, \Phi^\dagger_j)$ allowed by the symmetries of the theory.

(ii) The kinetic terms in $M_{\rm Planck} = 1$ units are given by

$$\mathcal{L}_{\rm kin} \;=\; G^i_j\,\partial^\mu\Phi^*_i\,\partial_\mu\Phi_j\,; \quad G^i_j = \frac{\partial^2 G}{\partial\Phi^*_i\partial\Phi_j}\,.$$

(iii) The effective potential V in units of $M_{\rm Planck}$ is

$$V \;=\; e^G\,\left[\,G_i\,(G^{-1})^i_j\,G^j \;-\; 3\,\right]\,; \quad G_i(G^i) = \frac{\partial G}{\partial\Phi_i(\Phi^*_i)}\,.$$

(iv) SUSY can be broken spontaneously by $G_i \neq 0$, while $\langle V\rangle = 0$. This can solve the so-called cosmological constant problem, although fine-tuning the tree-level potential is still required.

(v) The potential V is not always positive definite $V \geq 0$, and could have lower 'false' vacua. Hopefully, the tunneling time to those 'false' vacua is sufficient large, i.e. much larger than the age of our universe $\sim\; 10^{10}$ years.

– Gravity-Mediated SUSY Breaking

In this scenario, the breaking of SUSY occurs in the so-called hidden sector of the theory. This breaking gets communicated to the visible sector through non-renormalizable interactions of hidden sector fields, e.g. X, with the visible SM gauge fields. The strength of these interactions is *gravitationally* suppressed by powers of $1/M_{\rm Planck}$.

The generic form of these *gravity-mediated* interactions is

$$\begin{aligned} -\mathcal{L}_{\rm soft} \;=\; & \int d^4\theta \, \Big\{ \big(\frac{X}{M_{\rm P}}\, f\, W^\alpha W_\alpha\, \delta(\bar\theta) \;+\; {\rm h.c.}\big) \\ & + \frac{X^\dagger X}{M_{\rm P}^2}\, G^i_j \Phi^\dagger_i \Phi_j \\ & + \big[\, \delta(\bar\theta) \frac{X}{M_{\rm P}} \big(\frac{1}{2} b_{ij}\Phi_i\Phi_j \;+\; \frac{1}{6} a_{ijk}\Phi_i\Phi_j\Phi_k\big) \;+\; {\rm h.c.}\,\big] \Big\} \,. \end{aligned}$$

The spontaneous breaking of SUSY in the hidden sector is manifested by a non-vanishing VEV of the auxiliary field F_X of X, i.e. $\langle F_X \rangle = M^2_{\rm hidden} \neq 0$.

Exercises:

(i) Show that $\mathcal{L}_{\rm soft}$ given above is fully equivalent to the soft SUSY-breaking Lagrangian on page 63.

(ii) If the scale of SUSY breaking in the visible sector is $m_{3/2} \sim 1$ TeV to solve the gauge hierarchy problem, estimate the scale of SUSY breaking $M_{\rm hidden}$ in the hidden sector.