Fundamentals of Wakefields and Collective Effects:
From Physical-Mathematical Analysis to Practical Applications

Instructor: Roger M. Jones, University of Manchester/Cockcroft Institute

Purpose and Audience

The purpose of the course is to enable students to become well-versed in the beam dynamics of wakefield-beam interaction in high energy accelerators. It is suitable for advanced undergraduates, graduate students and, active researchers in the field. *Prerequisites: A course on Electromagnetism and a minimum of at least a mathematical background of first-year undergraduate calculus.*

Objectives

This course will address the fundamentals of wakefields and their relation to the beam impedance. The features of both long-range and short-range wakefields will be discussed. Circuit models of relativistic electron beams coupled to multiple accelerator cavities will be developed to calculate the coupled modal frequencies and wakefields. In addition to the general theoretical formalism of wakefields, practical methods to damp and measure the wakefields will be described with techniques taken from ongoing research on high-energy linacs (L-band and X-band linacs in particular). Throughout the course, basic physical principles such as superposition, energy conservation and causality will be emphasized.

Instructional Method

Details of the lecture schedule are posted on the web

Course Content

The progress of multiple bunches of electrons through a linear or circular accelerator gives rise to a trailing electromagnetic field. This wakefield can have catastrophic consequences if its progress is left unchecked as the beam can become unstable and develop a BBU (Beam Break Up) instability. This course discusses the beam dynamics issues associated with wakefields and means of damping the fields to acceptable levels. Examples are taken from the recent international next generation linear colliders damping schemes. Wakefield issues in storage rings will also be discussed.

Background Reading

“RF Linear Accelerators”, Wiley & Sons Publishers (1998), by Thomas Wangler
“RF Superconductivity for Accelerators”, Wiley Publishers (1998), by Hasan Padamsee, Jens Knobloch and Tom Hays
"Physics of Collective Beam Instabilities in High Energy Accelerators" (free pdf download : http://www.slac.stanford.edu/%7Eachao/wileybook.html) , Wiley & Sons Publishers (1993) by Alexander Chao
"The Physics of Particle Accelerators: An Introduction", Oxford University Press (2000) by Klaus Wille

"Fundamentals of Beam Physics" Oxford University Press (2003) by James Rosenzweig
“Particle Accelerator Physics I & II”, (study edition) Springer-Verlag (2003) by Helmut Wiedemann
“Impedances and Wakes in High Energy Particle Accelerators”, World Scientific Publishers (1998), by Bruno W Zotter and Semyon Kheifets

Credit Requirements

None.

2015 Cockcroft Institute Lecture Series:

Duration: June/July

FUNDAMENTALS OF WAKEFIELDS AND IMPEDANCE: FROM PHYSICAL-MATHEMATICAL ANALYSIS TO PRACTICAL APPLICATIONS

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University of Manchester/Cockcroft Institute

In this course, wakefields are analyzed and practical structures which limit emittance growth are demonstrated. This course will last for five 3 weeks. A familiarity with fundamental concepts of accelerator physics is assumed. Basic features of wakefields are outlined and detailed results on wakefield minimization and beam diagnostics based on the ILC (International Linear Collider) superconducting L-band linacs are described together with the features of X-band normal conducting linacs.

1. **Part I of Fundamentals of wakefields and impedance**: Basic concepts and definitions are introduced. A field function analysis of wakefields is discussed and practical simplifications are introduced. The features of short-range and long-range wakefields are sketched out.

2. **Part II of Fundamentals of wakefields and impedance and applications to linear colliders**. Further general features of wakefields are described. The wakes in both L-band (superconducting) and X-band (normal conducting) linacs are investigated. Mode coupling issues that are likely to arise in the ILC main superconducting linacs are described. A circuit model of the dipole wakefield is developed for moderate to heavily damped accelerator structures. Interleaving the cell frequencies of adjacent structures is introduced as a means to combat insufficient fall-off in wakefields. Manifold damped structures are modeled with a transmission-line combined with an L-C circuit model and the additional features (built-in BPM and structure alignment thorough monitoring of manifold radiation) of DDS (Damped Detuned Structures) are modeled in detail. This may have particular relevance to CLIC.

3. **Special topics**: Detailed study of resistive wall wake. BBU (Beam Break Up).

4. **Impedance and wakefield via a bench measurement**. Higher modes of the TESLA accelerator and measurements made at the TTF (TESLA Test Facilty). A coaxial wire method, for determining the modes likely to be excited by a particle beam, is described, from its original concept though to the latest research

5. **Assessment via tutorial problems**

OVERVIEW

- **The Transverse Wakefield Problem**
- **Wakefield Definitions**
- **Wakefield Examples and Methods of Calculation**
- **Wakefield Fundamentals - Panofsky-Wenzel Theorem**
- **Modal Sum Representation of Wakefield via Field Function Analysis**
- **R-L-C Circuit Model of Single Mode and Impedance-Wake Relations**
- **Pill-Box Wake Function**
- **Scattering from Sharp Transitions**

Introduction

- **In this course, we will focus on ultra relativistic beams. The quantity under consideration has a finite value as the particle velocity approaches the velocity of light –often we will use v = c.**

- **The bunch lengths under consideration are sub-millimeter and even tens of microns (ILC, LCLS II).**

- **I will try to maintain S.I units throughout –for the sake of consistency.**

- **Lectures: Monday, 10:30 to 12:30.**
  No computer labs planned as yet.

- **There will be some problem assignments for afternoon tutorial sessions.**

- **No examination!**

General Properties of Wakefields

- **Longitudinal wakes give rise to energy spread over the train of bunches being accelerated.**

- **Transverse wakes in high frequency linear accelerators (linacs), if left unchecked, can readily *dilute the emittance of the beam*.**

- **Can give rise to instability that causes the beam to oscillate transversely –**
  ***Beam Break Up (BBU) instability*.**

- **The instability that develops in a linac is a single pass instability.**
  - **In circular accelerators the effect is cumulative and the feedback mechanism amplifies the growth turn-by-turn. The growth is $\propto \exp(\Gamma t)$**
  - **We will analyze the growth effect in linacs.**

- **It is crucial to damp the wakefields such they are not an issue!**

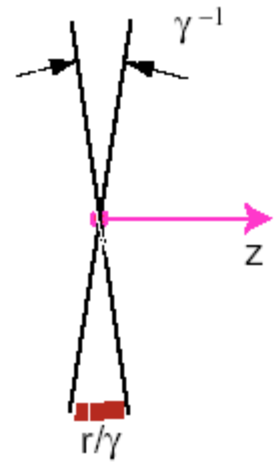
- **Or, why not try to make use of the trailing wakefield to accelerate beam? Plasma wakefield accelerators are a possibility! Plasma wakefield acceleration/focusing is ongoing research at SLAC/LBNL/CERN (FACET/BELLA/AWAKE)**

- **In order to optimize the cost of acceleration high energy linacs accelerate multiple bunches of electrons/positrons within an rf pulse train.**

Relativistic Point Charge in Free Space

- **A point at rest has an isotropic distribution of electric field**

- **Consider point charge moving in the z-direction**
- **For v~c then: γ>>1 and the field is squeezed in the longitudinal direction**



- **The field is limited to a "pancake-like" region and in the limit of v=c the field is entirely transverse (the pancake has zero angular spread)**
- **The field for a particle moving with constant velocity is given by:**

$$\mathbf{E}=\frac{q\mathbf{R}}{4\pi\varepsilon_0\gamma R_*^3},\quad H=Y_0\frac{\mathbf{v}}{c}\times\mathbf{E} \qquad \textbf{(1.1)}$$

where the vector R is drawn from the center of the charge q, to the observation point, $R_*^2=z^2+r^2/\gamma^2$**, and** $\gamma=\sqrt{1-v^2/c^2}$ **.**

$$E_r(z,r) = \frac{q\gamma r}{4\pi\varepsilon_0\left(z^2\gamma^2 + r^2\right)^{3/2}} \qquad \textbf{(1.2)}$$

- **In the pancake region, z~r/γ, or angle 1/γ:**

$$E_r = Z_0 H_\phi \sim \frac{\gamma q}{4\pi\varepsilon_0 r^2}, \quad E_z \sim \frac{q}{4\pi\varepsilon_0 r^2} \qquad \textbf{(1.3)}$$

- **No net power is transferred transverse to the particles motion, but there is a non-zero Poynting flux flowing parallel to the particle and attached to it**

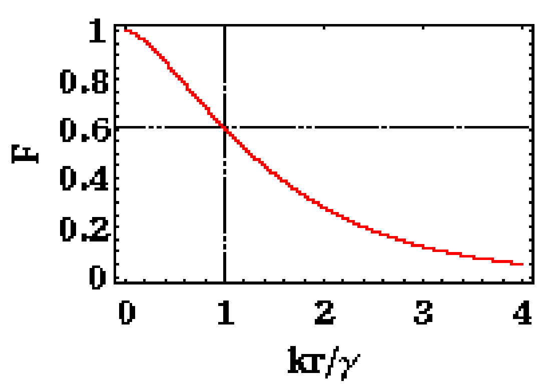
Plane Wave Fourier Decomposition

- **In the ultrarelativistic limit v->c the beam field is a plane wave electromagnetic field**
- **We decompose the field by means the Fourier transform:**

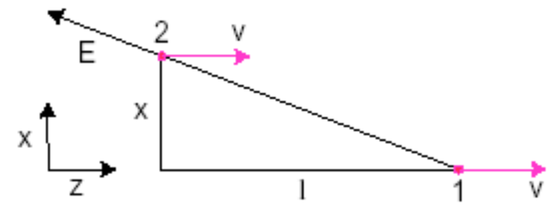
$$E_r(z,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dz\frac{q\gamma r}{4\pi\varepsilon_0\left(z^2\gamma^2+r^2\right)^{3/2}}e^{ikz}=\frac{qk}{4\pi^2\varepsilon_0\gamma r}K_1\left(\frac{kr}{\gamma}\right)=\frac{q}{4\pi^2\varepsilon_0 r^2}F\left(\frac{kr}{\gamma}\right)$$

(1.4)

- **The beam field is thus a superposition of plane waves with the spectrum F. The spectral width is:** $\Delta k \sim \gamma / r$



Space Charge Force



- **In practice the forces that result from the beam are more important than the detailed structure of the fields themselves**

- **We require the force the leading particle exerts on the trailing one**

- **The force exerted along the direction of motion of the beam, i.e. the longitudinal force is:**

$$F_l = qE_z = -\frac{q^2 l}{4\pi\varepsilon_0\gamma^2\left(l^2 + x^2/\gamma^2\right)^{3/2}}$$

- **The transverse force is:**

$$F_t = q(E_x - vB_y) = \frac{q^2 x}{4\pi\varepsilon_0\gamma^4\left(l^2 + x^2/\gamma^2\right)^{3/2}}$$

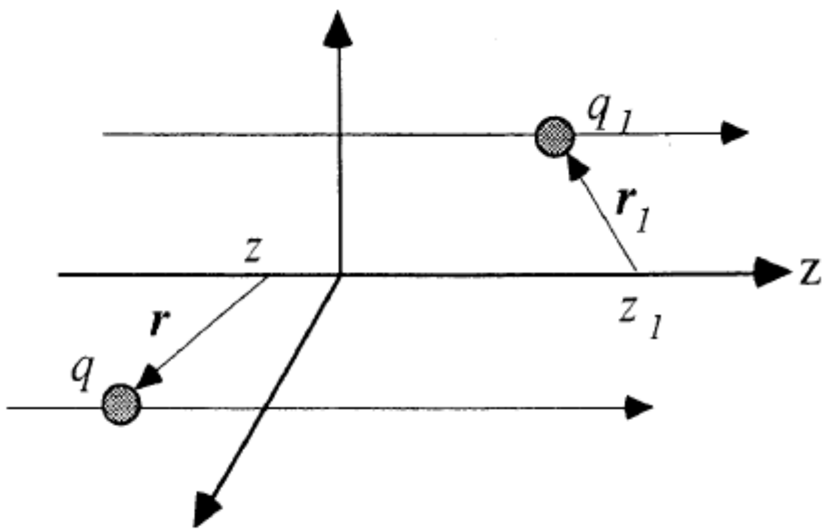
- **The longitudinal force decreases as γ^{-2} (for $l>x/\gamma$).**

- **The transverse force decays even more rapidly, for $l>x/\gamma$, $F_t \sim \gamma^{-4}$**

- **But for close proximity to the particle $l \sim 0$ we find $F_t \sim \gamma^{-1}$**

- **In all cases in the limit of v=c all forces on the particle vanish**

Longitudinal Wake function And Loss Factor



The energy lost by the charge q_1 is given by the work done by the longitudinal e.m. force along the structure:

$$U_{11} = -\int_{-\infty}^{\infty} F(\mathbf{r}_1, z_1, \mathbf{r}_1, z_1; t).dz = -q_1 \int_{-\infty}^{\infty} \left[\mathbf{E}(\mathbf{r}_1, z_1, \mathbf{r}_1, z_1; t) + \mathbf{v}\times\mathbf{B}(\mathbf{r}_1, z_1, \mathbf{r}_1, z_1; t)\right].dz \quad \textbf{(1.5)}$$

and this is evaluated at time $t=z_1/v$. The trailing charge changes its energy as it is influenced by field trailing the driving charge:

$$U_{21} = -\int_{-\infty}^{\infty} F(\mathbf{r}, z, \mathbf{r}_1, z_1; t).dz = -q \int_{-\infty}^{\infty} \left[\mathbf{E}(\mathbf{r}, z, \mathbf{r}_1, z_1; t) + \mathbf{v}\times\mathbf{B}(\mathbf{r}, z, \mathbf{r}_1, z_1; t)\right].dz \quad \textbf{(1.6)}$$

and this is evaluated at time $t=z_1/v+\tau$, where τ is the time delay between the driving and the witness bunch.

- **A physical accelerator (a linac or an accelerator ring) is not infinite in length!**
- **Providing the fields are confined within a given region and evanescent elsewhere then truncating the integral gives a very good approximation of the energy.**

- **A real vacuum chamber, standing wave accelerator, etc is formed by smooth transitions in the geometry and has various devices inserted such as RF cavities, kickers, diagnostic components, etc. These devices perturb the fields. Even with parallel computing, codes with relatively large amounts of memory one is never able to model the full set of accelerator components simultaneously.**

- **In modeling the energy losses and impedance one models the individual components and sums the losses. This is usually quite accurate unless significant modal distortion takes place.**

- **Circuit models can account for mode distortion in NLC accelerating structures for example (later lectures).**

The loss factor is defined as the energy lost by q_1 per unit charge squared:

$$k(\mathbf{r}_1) = \frac{U_{11}(\mathbf{r}_1)}{q_1^2} \qquad \textbf{(1.7)}$$

And the longitudinal wake can be defined in terms of the energy lost by the trailing charge per unit q_1 per unit q:

$$w_z(\mathbf{r},\mathbf{r}_1;\tau) = \frac{U_{11}(\mathbf{r},\mathbf{r}_1;\tau)}{q_1 q} \qquad \textbf{(1.8)}$$

Both the longitudinal wake and the loss factor have the same units, viz. Volts/Coulomb (V/C).

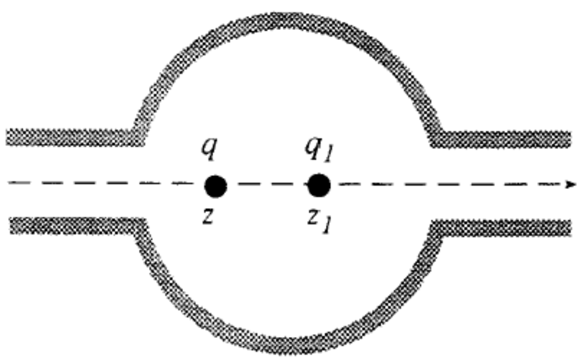
Often the wake per unit length is the practical quantity of interest (especially for periodic structures for example =>wake/unit period):

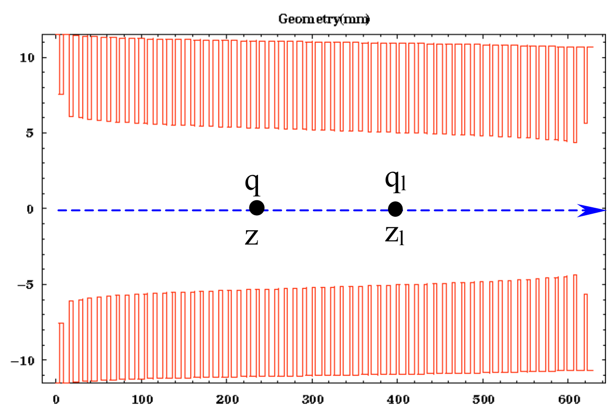
$$\frac{d}{dz} w_z(\mathbf{r},\mathbf{r}_1;\tau) = -\frac{1}{q_1 q} F_z(\mathbf{r},z,\mathbf{r}_1,z;\tau); \quad z = z_1 - v\tau$$

This wake has units of Volts/Coulomb/meter (c.f. the resistive wall impedance lecture #1 homework).

- **The wake function is, of course, no more than the force per unit charge acting on q.**

- **Important to note that in most practical cases the structures have some symmetry (circular, elliptical, rectangular) and the beam moves by a small amount from the electrical axis of symmetry.**



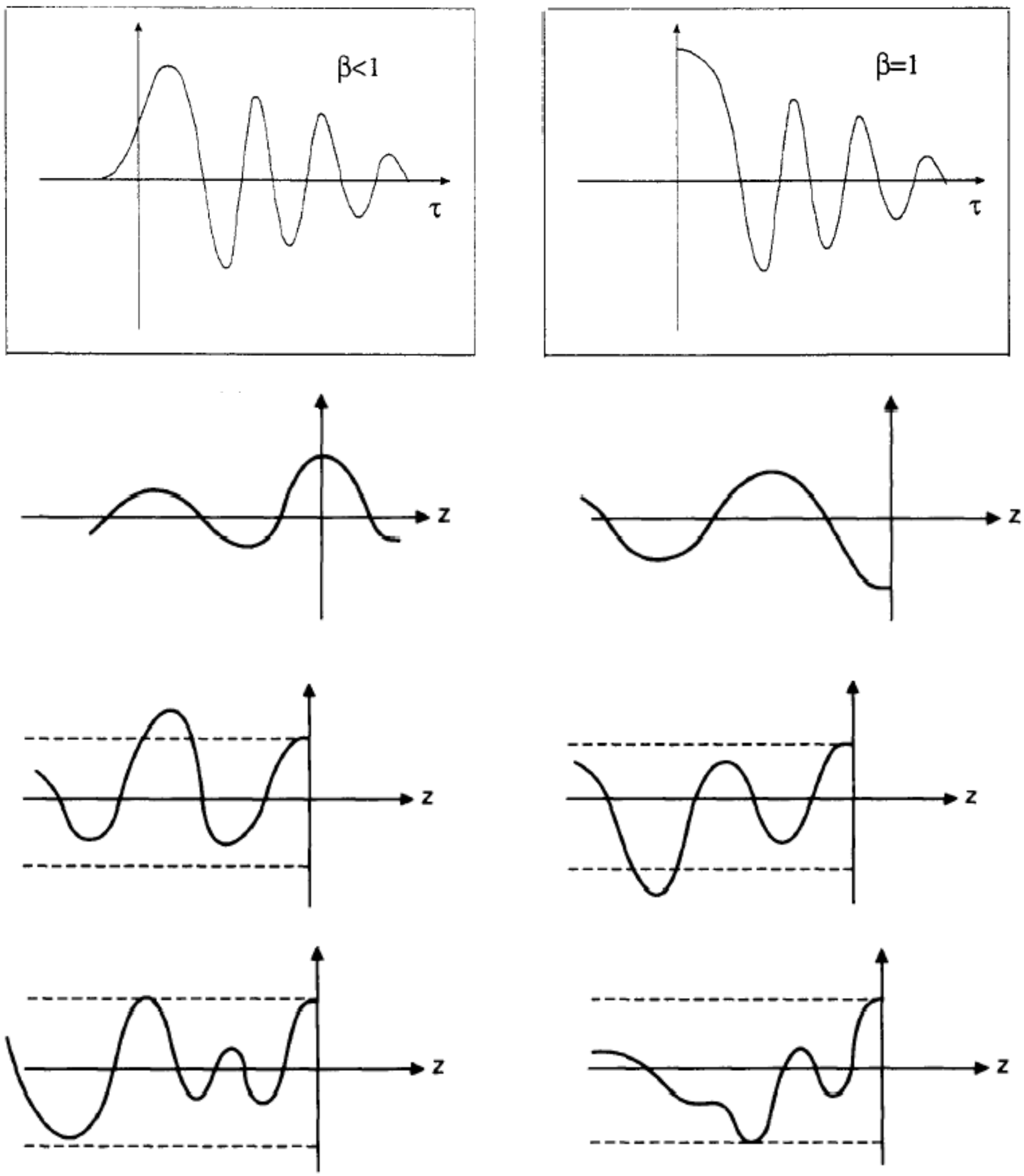


(a)

Leading and trailing charges in spherical cavity (a) and cylindrically symmetric circular cavities (b). The cell on the left is representative of a superconducting TESLA cell and on the right is a multi-cell NLC X-band accelerator.

- **This means that in a multi-pole expansion of fields only the first few terms will be significant.**
- **Typically only monopole and dipole terms are meaningful in realistic simulations**

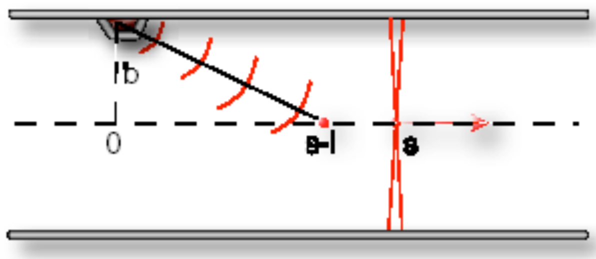
Physically Realizable Wakes?



- **All wakes, apart from the upper pair are non-physical!**
- **Wakes that are non-zero ahead of the particle break causality.**
- **The solitary case of β<1 is allowed to have Wz(s) finite for s<0**

  (the "pancake" of field has a finite width)

Characteristic "Catch-Up" Distance



- **In the limit v=c the field can only interact with the trailing particles. This is called the principle of causality.**

- **At v=c the field cannot precede the bunch**

- **Assume a discontinuity at s=0 scatters the field, and the leading particle passes this point at time t=0**

- **The scattered field reaches the point l behind the drive particle at time t and:** $ct=\sqrt{(s-l)^2+b^2}$**, where** $s(=vt)$ **is the coordinate of the leading particle at time t**

- **Assume** $l \ll b$ **and** $b \ll s$ **and thus:**

$$s=\sqrt{(s-l)^2+b^2}\sim s\left(1-\frac{2l}{s}+\frac{b^2}{s^2}\right)$$

- **The catch-up distance is then obtained:**

$$s \sim \frac{b^2}{2l}$$

- **Typically the length l is of the order of the bunch length: $l \sim \sigma_z$**

- **For example for the b=4mm and $\sigma_z \sim 100\ \mu m$:**

$$s \sim \frac{b^2}{2l} \sim 8\text{cm}$$

- **Thus, in simulating the wake with a code such as ABCI, then it is important to ensure the simulation length is larger than the catch-up distance**

Longitudinal Wake Function and Loss Function of Bunch Distribution

- **The wake functions and loss factors we have defined have been for point charge, i.e. the wake function is a Green's function.**

- **Convolution of the Green's function with the actual current distribution gives the true wake function:**

$$W_z(\tau) = \frac{U(r,\tau)}{q_1 q} = \frac{1}{q_1}\int_{-\infty}^{\tau} i_b(\tau') w_z(\mathbf{r}, \tau - \tau').d\tau' \qquad \textbf{(1.9)}$$

- **By superposition we also obtain:**

$$K(\mathbf{r}) = \frac{U(r,\tau)}{q_1^2} = \frac{1}{q_1}\int_{-\infty}^{\tau} W_z(r,\tau) i_b(\tau) d\tau \qquad \textbf{(1.10)}$$

For example, a rectangular bunch distribution, for a delta function wake, is readily integrated. The bunch current and point wake are of the form:

$$i_b(t) = q_1 \frac{H[t+T] - H[t-T]}{2T} \qquad \textbf{(1.11)}$$

$$w_z(\tau) = w_0 \cos[\omega_r \tau] H[\tau] \qquad \textbf{(1.12)}$$

In the region of the bunch we obtain (-T<τ<T):

$$W_z(\tau) = \frac{w_0}{2} \frac{\mathrm{Sin}\left[\omega_r\left(\tau+T\right)\right]}{\omega_r T} H\left[\tau+T\right] \quad \textbf{(1.13)}$$

In the limit of T->0 we obtain a delta function $i_q(\tau) = q_l$ $\delta(\tau)$ and thus the point source wake is uncovered: $W_z(\tau)$->$w_z(\tau)$

Also, setting τ=0 in $W_z(\tau)$ and taking the limit of T->0:

$$\lim_{T->0} W_z(0) = \frac{w_0}{2} \quad \textbf{(1.14)}$$

Thus, the wake at the bunch is half that of the total wake function. The bunch loss factor is also obtained:

$$K = \frac{w_0}{2}\left[\frac{\mathrm{Sin}\left(\omega_r T\right)}{\omega_r T}\right]^2 \quad \textbf{(1.15)}$$

In the limit of T->0 we obtain the point charge loss factor:

$$k = \lim_{T->0} K = \frac{w_0}{2} \quad \textbf{(1.16)}$$

The wake function external to the distribution, i.e. for τ>=T:

$$W_z(\tau) = w_0 \frac{\mathrm{Sin}\left(\omega_r T\right)\cos\left(\omega_r \tau\right)}{\omega_r T} H\left[\tau-T\right] \quad \textbf{(1.17)}$$

Thus, in the limit of T->0 and τ->0 in (1.17)

Longitudinal Coupling Impedance

The impedance is given by spectrum of the longitudinal point charge wake function:

$$Z_z(r,r_1;\omega) = \int_{-\infty}^{\infty} w_z(r,r_1;\tau)\exp(-j\omega\tau)d\tau \qquad \textbf{(1.18)}$$

and the point charge wake is given by the inverse Fourier transform of the impedance:

$$w_z(r,r_1;\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_z(r,r_1;\omega)\exp(j\omega\tau)d\omega \qquad \textbf{(1.19)}$$

The Fourier transform possesses real and imaginary parts and they are related by the Kramers-Kronig, or Hilbert transform.

Also, as we have seen in the case of the rectangular distribution:

$$k = W_z(r,r_1;0)/2 \qquad \textbf{(1.20)}$$

and in terms of impedance :

$$k = W_z(r,r_1;0)/2 = \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}\{Z_z(r,r_1;\omega)\}d\omega \qquad \textbf{(1.21)}$$

Thus, we can think of the real part of the impedance as the power spectrum of the energy loss. This can be generalized to the complex impedance being related to the complex power spectrum of the energy loss.

In practice a Gaussian profile is often used:

$$I(t) = q_1 c \frac{\exp(-\frac{c^2 t^2}{2\sigma^2})}{\sqrt{2\pi}\sigma}, \qquad I(\omega) = q_1 \exp(-\frac{\omega^2 \sigma^2}{2c^2}) \qquad \textbf{(1.22)}$$

The current is normalized such that: $\int_{-\infty}^{\infty} I(t)dt = q_1$

Wakefunctions, Bunch Distributions and Impedance

- **The objective is to obtain the total bunch wake in terms of the current and impedance**

Recall the fact that the wake due to a distribution is:

$$W_z(\tau) = \frac{U(r,\tau)}{q_1 q} = \frac{1}{q_1}\int_{-\infty}^{\tau} i_b(\tau') w_z(\mathbf{r}, \tau - \tau') . d\tau'$$

and take the Fourier transform:

$$W_z(\tau) = \frac{U(r;\tau)}{q_1 q} = \frac{1}{2\pi q_1}\int_{-\infty}^{\infty} Z(r;\omega) I(\omega) \exp(j\omega\tau) d\omega \qquad \textbf{(1.23)}$$

The loss factor:

$$K(\mathbf{r}) = \frac{U(r,\tau)}{q_1^2} = \frac{1}{q_1}\int_{-\infty}^{\tau} W_z(r,\tau) i_b(\tau) d\tau$$

Again, performing a Fourier transform gives:

$$K = \frac{U(r)}{q_1^2} = \frac{1}{\pi}\int_{0}^{\infty} Z(r;\omega) \left|I(\omega)\right|^2 d\omega \qquad \textbf{(1.24)}$$

For a Gaussian beam we have:

$$K(r) = \frac{1}{\pi}\int_{0}^{\infty} Z(r;\omega) \exp\left(-\frac{\omega^2\sigma^2}{c^2}\right) d\omega \qquad \textbf{(1.25)}$$

- **As σ->0 then the current goes to infinity and, as expected, the loss factor (in (1.25)) becomes that of the point source k.**

For example, taking the case of a parallel R-L-C circuit the impedance is given by:

$$Z_1(\omega) = \frac{2k/\omega}{\omega_0/(Q\omega) - I(\omega_0/\omega - 1)(\omega_0/\omega + 1)} \qquad \textbf{(1.26)}$$

where the losses are represented by the quality factor Q ($=\omega_0 RC$)**. The bunch wake and loss factors are quite complicated expressions for this case. For low losses, the impedance simplifies to:**

$$Z_1(\omega) = 2I\frac{k\omega}{\omega_0^2 - \omega^2} \qquad \textbf{(1.27)}$$

The bunch wake function, using (1.23) is now:

$$W_z(\tau) = 2ke^{-\frac{\sigma^2\omega_0^2}{2c^2}}\cos\omega_0\tau \qquad \textbf{(1.28)}$$

and the bunch loss factor for a Gaussian beam, using (1.25), becomes:

$$K = ke^{-\frac{\sigma^2\omega_0^2}{c^2}} \qquad \textbf{(1.29)}$$

Both of these results ((1.28) and (1.29)) are only valid for τ>3σ because this allows the infinite limits to be

used in the convolution of the bunch distribution with the point source wake.

Note that for an R-L-C parallel resonance circuit the loss factor can be written:

$$k = \frac{\omega_0 R}{2Q} \qquad \textbf{(1.30)}$$

In accelerator physics the shunt impedance is usually defined such that a factor of 4 (rather than 2) occurs in the denominator of (1.30) –the context should make it clear as to whether or not 4 is used.

- **The loss factor is, in general a function of the r.m.s. length of the bunch. For Gaussian bunches, in general one finds:**

$$Z_r(\omega) \propto \omega^a \Leftrightarrow K \propto \sigma_t^{-(a+1)}$$

- **Note the "r" dependence may be dropped as it will be understood to be present according to the context of the wake function and impedance.**

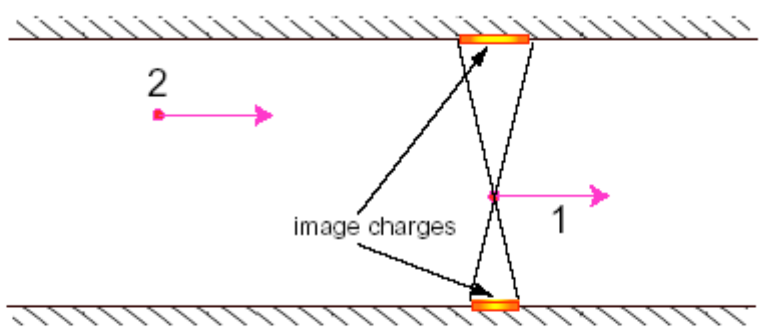
END OF LECTURE 1

Wakefield For Perfectly Conducting Structures

- **For ultrarelativistic beams in perfectly conducting accelerator structures the longitudinal and transverse forces on a beam vanish**
  - **No wakefield in limit β (= v/c)->1 (γ->infinity)**
  - **Only true with no obstacles to reflect the field**

- **Why?**

  - **A particle traveling in a perfectly conducting cylindrical pipe induces image charges on the surface of the wall. These image charges travel with the same velocity c.**

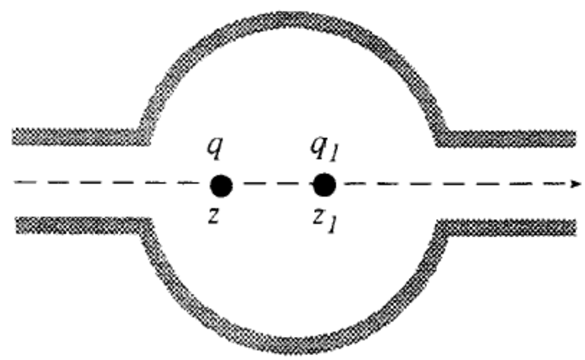


  - **Since the particle and image charge move on parallel paths, in the limit of v=c they do not interact with each other.**

Longitudinal Wake as a Summation of Multipoles

- **For accelerators and microwave components with cylindrical symmetry it is natural to assume that the wake functions can be expanded over modes exhibiting the symmetry**

- **We consider charges moving on axis.**
- **The coordinates of the driving charge and witness charge are ($r_l, \phi_l=0, z_l$) and ($r, \phi=0, z$) respectively.**



- **The charge density can be represented as a superposition of multipole moments in cyclindrical coordinates:**

$$\rho_l = q_l \frac{\delta(r-r_l)}{r_l}\delta(\phi)\delta(z-z_l) = \frac{q_l}{2\pi}\frac{\delta(r-r_l)}{r_l}\delta(z-z_l)\sum_{m=0}^{\infty}\left(2-\delta_m^0\right)\cos m\phi \quad \textbf{(1.31)}$$

- **Where the azimuthal symmetry of the geometry has been utilized and $\delta_m^0=0$ for m>0, $\delta_0^0=1$, and $z_l = \beta c\tau$.**

- **Thus, the charge can be envisaged the summation of a series of charged rings with angular dependence cos(mϕ).**
- **m=0 for example represents a charges ring with uniform density up to r=r$_1$**

- **The wake is nothing more than the solution to Maxwell's equations with a charge source driving the differential equations and hence we make a superposition of multipole moments:**

$$w_z(\mathbf{r},\mathbf{r}_1;\tau) = \sum_{m=0}^{\infty} w_{z,m}(\mathbf{r},\mathbf{r}_1;\tau) = \sum_{m=0}^{\infty} \bar{w}_{z,m}(\mathbf{r},\mathbf{r}_1;\tau)\cos m\phi \qquad \textbf{(1.32)}$$

Radial Expansion of Wake Function in the Ultra-relativistic Limit

- **The e.m. fields produced by charges traveling down an accelerator structure are driven solutions of Maxwell's equations subject to the boundary conditions imposed at the walls.**

- **The longitudinal electric field is produced by the bunch of charged particles, and by the currents induced in the walls. Considering only the induced fields, it can shown that**

$$\left[\nabla_{\perp}^{2}-\left(\frac{\omega}{\beta\gamma c}\right)^{2}\right]\tilde{E}_{z}=0 \qquad \textbf{(1.33)}$$

- **In the ultrarelativistic limit (v=c) we clearly have:**

$$\nabla_{\perp}^{2}\tilde{E}_{z}=0 \qquad \textbf{(1.34)}$$

- **The solution in cylindrical coordinates is:**

$$w_{z}(r,r_{1};\tau)=\sum_{m=0}^{\infty}\bar{w}_{z,m}(r,r_{1};\tau)=\sum_{m=0}^{\infty}r^{m}r_{1}^{m}\bar{\bar{w}}_{z,m}(\tau) \qquad \textbf{(1.35)}$$

- **The monopole mode, m=0, does not depend on radial position. Use is often made of this fact when calculating the wake by placing the evaluation**

point at the radius of the beam tube where the field is zero.

- **The expansion concerns the secondary fields induced by the beam. The space charge fields show a different dependence.**

Multipole Longitudinal Impedance

- **As the impedance is no more than the Fourier transform of the wake function then it too can be expanded in a multipole expansion:**

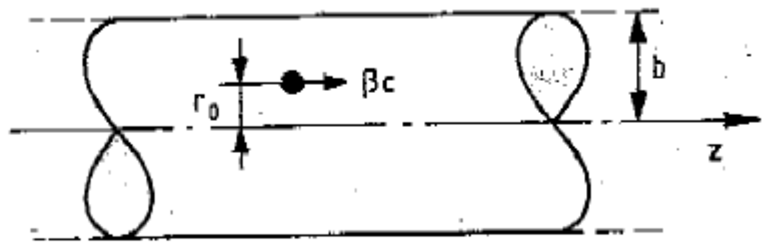
$$Z(r,r_1;\omega) = \sum_{m=0}^{\infty} Z_m(r,r_1;\tau) = \sum_{m=0}^{\infty} \bar{Z}_m(r,r_1;\tau)\cos m\phi \qquad \textbf{(1.36)}$$

- **For ultra relativistic charges the radial dependence is known:**

$$\bar{Z}_m(r,r_1;\omega) = r^m r_1^m \bar{\bar{Z}}_m(\omega) \qquad \textbf{(1.37)}$$

where $\bar{\bar{Z}}_m$ has dimensions Ω/m^{2m}

Analysis of Wake in Cylindrically Symmetric Circular Waveguide



We will analyze the wakefield that occurs in a cylindrically symmetric circular waveguide. The e-m fields are written in terms of vector potential and electrostatic potential:

$$\mathbf{E} = -\dot{\mathbf{A}} - \nabla V, \ \mathbf{B} = \nabla x \mathbf{A} \quad (1.38)$$

and this enables Maxwell's equations to be written:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) V = -\frac{\rho}{\varepsilon_0} \quad (1.39)$$

the relation between vector and scalar potential are not uniquely determined and thus we are free to impose the following relation, the Lorentz condition:

$$\nabla.\mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \quad (1.40)$$

Making the transformation of variables u = z - βct we find:

$$\nabla_t^2 V - \nabla_u^2 V - \beta^2 \nabla_u^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla_t^2 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial u^2}\right) V = -\frac{\rho}{\varepsilon_0} \quad \textbf{(1.41)}$$

and the electric field becomes:

$$\mathbf{E} = -\nabla_t V - \frac{\partial V}{\partial u}\hat{z} + \beta^2 \frac{\partial V}{\partial u}\hat{z}$$

$$\mathbf{E} = -\nabla_t V - \frac{1}{\gamma^2}\frac{\partial V}{\partial u}\hat{z} \quad \textbf{(1.42)}$$

For unit charge source (1.41) becomes:

$$\left(\nabla_t^2 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial u^2}\right) g(z,r,\theta) = -\frac{1}{\varepsilon_0}\frac{\delta(r-r_0)}{r_0}\delta(\theta-\theta_0)\delta(z-\beta ct) \quad \textbf{(1.43)}$$

The delta function is given in terms of the inverse Fourier transform of unity and in terms of a Fourier series:

$$\delta(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikz)dk$$
$$\delta(\theta-\theta_0) = \sum_{-\infty}^{\infty}(2-\delta_m^0)\cos[m(\theta-\theta_0)] \quad \textbf{(1.44)}$$

where $\delta_m^0 = \begin{cases} 1 & m=0 \\ 0 & m \neq 0 \end{cases}$

Thus the potential is expanded as:

$$g(z,r,\theta) = \sum_{-\infty}^{\infty} g_m(z,r)\cos[m(\theta-\theta_0)] \quad \textbf{(1.45)}$$

where:

$$g_m(z,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}_m^0(k,r)\exp(ikz)dk$$

Set t=0 in (1.43) and utilize the Fourier expansions given above:

$$\begin{aligned}&\left(\nabla_t^2 + \frac{1}{\gamma^2}\frac{\partial^2}{\partial u^2}\right)\sum_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}_m^0(k,r)\exp(ikz)dk\cos[m(\theta-\theta_0)] = \\ &-\frac{1}{2\pi\varepsilon_0 r_0}\int_{-\infty}^{\infty}\exp(ikz)dk\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left(2-\delta_m^0\right)\cos[m(\theta-\theta_0)]\end{aligned} \qquad \textbf{(1.46)}$$

Multiply the above by exp(-ik'z) and integrate with respect to z:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - q^2 - \frac{m^2}{r^2}\right)\tilde{G}_m^0(k,r) = -\frac{\delta(r-r_0)}{2\pi\varepsilon_0 r_0}(2-\delta_m^0) \qquad \textbf{(1.47)}$$

where q=k/γ. After some algebra the solution to the Green's function solution to this equation is written:

$$\tilde{G}_m^0(k,r) = -\frac{(2-\delta_m^0)}{2\pi\varepsilon_0}\begin{cases}K_m(qr_0)I_m(qr) & r<r_0 \\ K_m(qr)I_m(qr_0) & r>r_0\end{cases} \qquad \textbf{(1.48)}$$

where I and K are Bessel functions of the first and second kind, respectively. It is required that the potential is zero on the wall of the waveguide ($\tilde{G}_m^0(k,b)=0$). Thus, a term which accounts for image currents must be added to the Green's function:

$$\tilde{G}^{0}_{m}(k,r) = -\frac{(2-\delta^{0}_{m})}{2\pi\varepsilon_{0}}\left\{\begin{matrix} K_{m}(qr_{0})I_{m}(qr) \\ K_{m}(qr)I_{m}(qr_{0}) \end{matrix}\right. + B_{m}I_{m}(qr) \qquad \textbf{(1.49)}$$

and using $\tilde{G}^{0}_{m}(k,b)$=0 we find:

$$B_{m} = -\frac{I_{m}(qr_{0})}{I_{m}(qb)}K_{m}(qb) \qquad \textbf{(1.50)}$$

For a charge traveling on axis ($r_0 = 0$):

$$\tilde{G}^{0}_{m}(k,r) = -\frac{1}{2\pi\varepsilon_{0}}\left[K_{0}(qr) - \frac{I_{0}(qr)}{I_{0}(qb)}K_{0}(qb)\right] \qquad \textbf{(1.51)}$$

To avoid the divergence of the above function which occurs at r = 0 we convolve the Green's function over a disk of radius a and for q<<1 (low freq and/or high energy beams). For q<<1 the Bessel functions have a logarithmic dependence:

$$\tilde{G}^{0}_{m}(k,r) \approx -\frac{1}{2\pi\varepsilon_{0}}\ln(r/b) \qquad \textbf{(1.52)}$$

Taking the convolution over a disk of charge of radius a:

$$V(k) = \frac{q}{4\pi\varepsilon_{0}}\left(1+\ln(b/a)\right) \qquad \textbf{(1.53)}$$

Also:

$$\tilde{E}(k) = -\frac{ik}{\gamma^{2}}V(k) \qquad \textbf{(1.54)}$$

Now, the longitudinal wave impedance is given by:

$$Z(\omega) = \frac{\tilde{E}(k)}{\beta c Q} = \frac{i\omega Z_0}{4\pi c(\beta\gamma)^2}\left[1 + 2\ln(b/a)\right] \qquad \textbf{(1.55)}$$

Where k = ω/(βc) has been used. This impedance is purely reactive, as would be expected, since there is no energy loss during the motion.

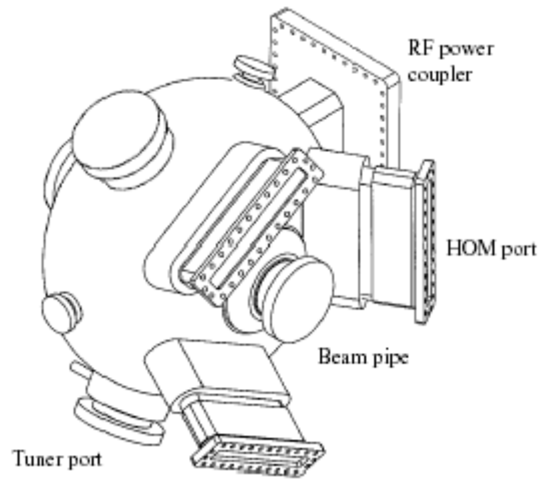
To return to (1.43) in the limit of an ultra-relativistic beam we obtain (1.52) and the inverse Fourier transform gives:

$$\left.\begin{aligned}
&g(r,z,t) = -\frac{1}{2\pi\varepsilon_0}\ln(r/b)\delta(z-\beta ct) \\
&E_z = O(\frac{1}{\gamma^2}) \\
&E_r = \frac{1}{2\pi\varepsilon_0 r}\delta(z-\beta ct) \\
&H_\phi = \frac{1}{2\pi r Z_0}\delta(z-\beta ct)
\end{aligned}\right\} \qquad \textbf{(1.56)}$$

i.e. in the limit of β->1 the fields are indistinguishable from those in free space. All the field components go to zero both behind the charge and ahead of the charge: there are no net forces on the charge. The wake potential vanished as $1/\gamma^2$: $W(\tau)=g(\tau)/\gamma^2$, τ=z-ct.

Synchronous Beam Fields

- **We have seen that the fields scattered from the obstacles (HOM ports, couplers, tuners, kickers, etc) give rise to non-zero wake functions.**
- **Some of these fields are localized around the bunch (resistive wall for example), others are localized in resonant structures such as the r.f. cavities.**
- **All these fields interact with the beam**
- ***Only the fields synchronous with the charges can change the energy of the charges***



PEP II Cavity: Beam pipe, HOM and tuner ports are illustrated

Making an expansion of the longitudinal field in a series of plane waves:

$$E_z(z,t) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\kappa \tilde{E}_z(\omega,\kappa) e^{j(\omega t - \kappa z)} \qquad \textbf{(1.57)}$$

- **The explicit dependence on (r, r_l, z_l) has been omitted**

In terms of the wake function we have:

$$w_z(\tau) = -\frac{1}{q_1}\int_{-\infty}^{\infty} E_z(z, t = \frac{z}{v} + \tau)dz =$$
$$-\left(\frac{1}{2\pi}\right)^2 \frac{1}{q_1}\int_{-\infty}^{\infty} e^{j\omega\tau}d\omega \int_{-\infty}^{\infty} d\kappa \tilde{E}_z(\omega, \kappa)\int_{-\infty}^{\infty} e^{-jz(\kappa-\kappa_0)}dz \tag{1.58}$$

where $\kappa_0 = \omega / v$. The point source delta function is given by:

$$\delta(\kappa - \kappa_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jz(\kappa-\kappa_0)}dz \tag{1.59}$$

Thus, only those components of the fields propagating with the same phase velocity can effect the charges energy. The fields from all other phases do not contribute as they average out to zero. We are left with:

$$w_z(\tau) = -\frac{1}{2\pi q_1}\int_{-\infty}^{\infty} e^{j\omega\tau}\tilde{E}_z(\omega, \kappa = \kappa_0)d\omega \tag{1.60}$$

Now, as the wake function is defined in terms of the longitudinal coupling impedance is defined as:

$$w_z(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_z(\omega)\exp(j\omega\tau)d\omega \tag{1.61}$$

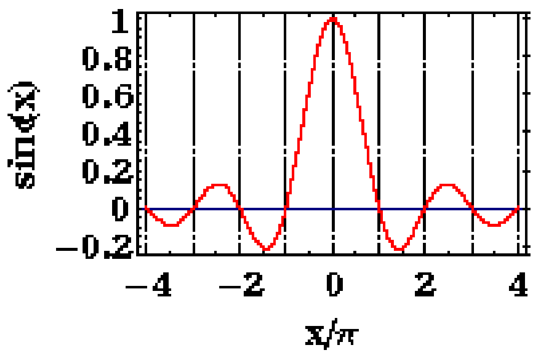
then it is clear that the impedance may also be written in terms of the Fourier transform of the field:

$$Z_z(\omega) = -\frac{1}{q_1}\tilde{E}_z(\kappa = \kappa_0, \omega) \tag{1.62}$$

For finite length cavities the delta function is replaced by the sinc function:

$$\frac{1}{2\pi}\int_{-L/2}^{L/2} e^{-jz(\kappa-\kappa_0)}dz = \frac{L}{2\pi}\frac{\sin\left[\left(\kappa-\kappa_0\right)\frac{L}{2}\right]}{\left(\kappa-\kappa_0\right)\frac{L}{2}} \qquad \textbf{(1.63)}$$

and the sinc becomes a delta function as the length L->infinity. For a finite a length, the sinc (=sin[x]/x) has a maximum at $\kappa=\kappa_0$ **and the zeros are located at** $\kappa=\kappa_0 \pm 2\pi/L$**.**



- **For long wavelengths the fields tend to be confined to a given region in which they propagate and are non-propagating (evanescent) elsewhere.**

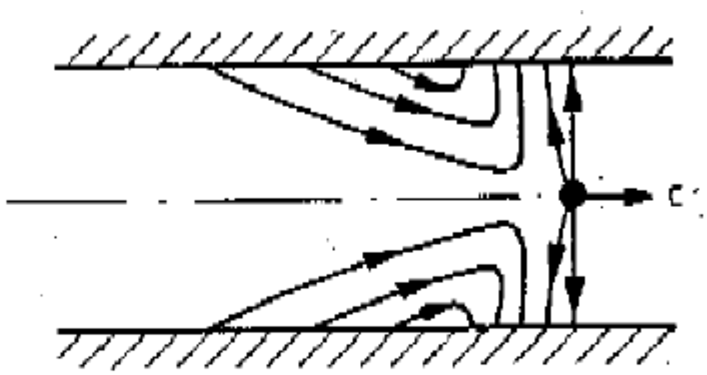
- **The integration path is confined to the propagating region.**

- **For short wavelengths fields propagate out of cavities into the beam tubes. However, the sinc function (1.63) shows that the contribution is small and hence the integration may be extended to infinite limits.**

Wakefield For Waveguide with Lossy Walls

- **For a pipe with finite conductivity σ and if the skin depth is much smaller than the thickness of the pipe wall then all of the field is essentially contained within a skin depth or so.**

- **Thus, the pipe walls can be considered to be infinite.**



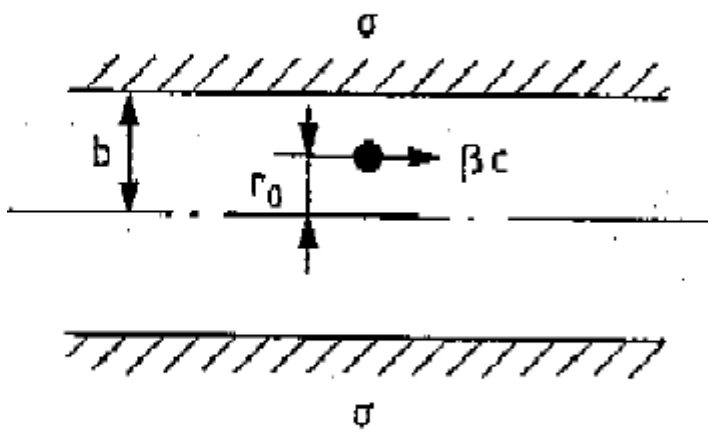
- **During the motion of the charged particle bunch a non-zero wakefield will appear behind the charge.**

- **In general the wakefield that develops from reflections from waveguide discontinuities (obstacles, tuners and irises etc) is far larger than the resistive wall wake.**
  - **However, for a collimator or beam scraper the resistive wall wake can be dominant effect. The collimator is used to scrape any beam halo that will develop on accelerating a relativistic beam through several km.**

Resistive Wall Wakefield

For a perfectly conducting matched waveguide with no obstacles to reflect back the field there is no overall wakefield. However, the presence of loss on the walls of the waveguide gives rise to a wakefield. For a cylindrical waveguide the E and H fields are given by:

$$\left.\begin{aligned} E_r(\omega) &= \frac{qZ_0}{2\pi r}e^{-jkz} \\ H_\phi(\omega) &= \frac{q}{2\pi r}e^{-jkz} \end{aligned}\right\} \qquad \textbf{(1.64)}$$



The continuity at the wall of the waveguide at r = b requires the magnetic field component inside the surface be the same as that outside. Inside the wall the field is sustained by a surface current flowing along the z direction (the waveguide is orientated along z). The electric field along the z-axis is given by:

$$E_z(\omega) = Z_c H_\phi(\omega) \qquad \textbf{(1.65)}$$

where the surface impedance is given by:

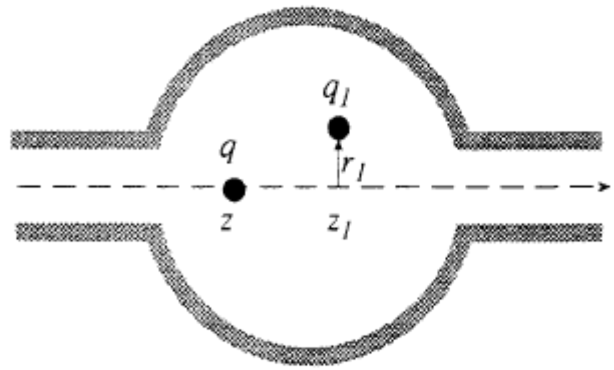
$$Z_c = \sqrt{\frac{j\omega Z_0}{\sigma c}} \qquad \textbf{(1.66)}$$

Z_0 (= 377) is the characteristic impedance of free space and c is the velocity of light. The flux of the Poynting vector on wall of the pipe gives the longitudinal impedance per unit length:

$$Z(\omega) = 2\pi b Z_c \left|H_\phi\right|^2 = \frac{1+j}{2\pi b}\sqrt{\frac{\omega Z_0}{2c\sigma}} \qquad \textbf{(1.67)}$$

- **This is generally a small effect compared to that encountered due to impedance of obstacles encountered in the accelerator structure. However, for short bunches with high charge considerable power dissipation may occur for non-superconducting cavities.**
- **The collimators for the ILC may have a significant resistive wall wakefield component as the energetic beam impinging on the walls of the collimator changes the conductivity. Experiments have been conducted in this area and are ongoing at SLAC with Cockcrofters participating.**

Transverse Wake Function



- **Drive charge, q_l, is displaced with respect to the axis of the cavity**
- **Multipole components are exited in the transverse plane: dipole, quadrupole, sextupole etc**
- **Trailing charge q is subject to a Lorentz force which has both longitudinal and transverse components**
- **Transverse momentum kick imparted to trailing charges:**

$$M_{21}(r,r_l;\tau) = \int_{-\infty}^{\infty} F_{\perp}(\mathbf{r},z,\mathbf{r}_l,z_l;t)dz, \quad t = \frac{z_l}{v} + \tau \qquad \textbf{(1.68)}$$

- **The integration is assumed to be over an infinite distance.**
- **A transverse displacement can lead to both vertical and horizontal kicks**

- **Transverse kick measured in Volts/Coulomb defines the transverse wake function:**

$$w_{\perp}(r,r_l;\tau) = \frac{M_{21}(r,r_l;\tau)}{q_l q} \qquad \textbf{(1.69)}$$

- **The dipole transverse loss factor is defined as the amplitude of the transverse momentum kick given to the charge by its own wake per unit charge (V/C):**

$$k_{\perp}(r) = \frac{M_{11}(r_1)}{q_1^2} \qquad \textbf{(1.70)}$$

- **The dipole component of the transverse kick is the dominant term for ultra relativistic charges. The *transverse dipole wake function* is defined as the transverse wake per unit of transverse displacement (V/Cm):**

$$w'_{\perp}(r, r_1; \tau) = \frac{M_{21}(r, r_1; \tau)}{q_1 q r_1} \qquad \textbf{(1.71)}$$

- **and the transverse loss factor (V/Cm):**

$$k'_{\perp}(r) = \frac{M_{11}(r_1)}{q_1^2 r_1} \qquad \textbf{(1.72)}$$

Transverse Wake Function and Loss Factor of a Bunch

As in the case of the longitudinal wake we take the convolution of the bunch current with the point source transverse wake function to obtain the bunch dependent wake:

$$W_{\perp}(\tau)=\frac{1}{q_1}\int_{-\infty}^{\infty} i_b(\tau')w_{\perp}(\mathbf{r},\tau-\tau').d\tau' \qquad \textbf{(1.73)}$$

and the bunch transverse loss factor :

$$K_{\perp}(\tau)=\frac{1}{q_1}\int_{-\infty}^{\infty} i_b(\tau)W_{\perp}(r;\tau)d\tau \qquad \textbf{(1.74)}$$

The transverse wake and loss factor per unit displacement are:

$$W_{\perp}'(r;\tau)=\frac{W_{\perp}(r;\tau)}{r} \qquad \textbf{(1.75)}$$

$$K_{\perp}'(r;\tau)=\frac{K_{\perp}(r;\tau)}{r} \qquad \textbf{(1.76)}$$

both of which are measured in Volt/Coulomb/meter

Panofsky-Wenzel Theorem Relating Longitudinal and Transverse Wakes

- **Firstly, we consider both the driving charge and the witness charge both lying moving along the z-axis of the accelerator.**

- **Once the longitudinal component of the wakefield has been calculated the transverse wakefield can be derived from it in a straightforward manner. From Maxwell's equation:** $\nabla \mathrm{x} \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$ **then:**

$$\mathbf{e}_z \mathrm{x} \frac{\partial}{\partial t}\mathbf{B} = \frac{\partial}{\partial z}\mathbf{E}_{\perp} - \nabla_{\perp} E_z \quad \textbf{(1.77)}$$

- **Using the relation for the total derivative:**

$$\frac{d}{dz}\mathbf{E}_{\perp}(r, r_1, (z+s)/c) = \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{E}_{\perp}(r, r_1, (z+s)/c) \quad \textbf{(1.78)}$$

- **then the derivative of the transverse wake with respect to s is written in the form:**

$$\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r, r_1, (z+s)/c) = \frac{1}{q_1}\int_{-\infty}^{\infty} dz \left[\frac{d}{dz}\mathbf{E}_{\perp}\left(r, r_1, (z+s)/c\right) - \nabla_{\perp} E_z\left(r, r_1, (z+s)/c\right)\right] \quad \textbf{(1.79)}$$

- **Performing the integrals as indicated above the first term vanishes provided $\mathbf{E}_\perp$ vanishes at the boundaries and we are left with (V/C/m):**

$$\frac{\partial}{\partial s}\mathbf{W}_\perp(r,r_1,s) = -\nabla_\perp W_P(r,r_1,s) \qquad \textbf{(1.80)}$$

- **This is the *Panofsky Wenzel* theorem (1956). A single integration provides the transverse wakefield once the longitudinal has been calculated:**

$$\mathbf{W}_\perp(r,r_1,s) = -\nabla_\perp \int_{-\infty}^{s} W_P(r,r_1,s')ds' \qquad \textbf{(1.81)}$$

- **In applying the above formula it has been assumed that $\lim_{s->\infty}\mathbf{W}_\perp(r,r_1,s)=0$. In practice one has finite limits and the lower limit is often taken at a point in which the field is zero (on the walls of a perfectly conductor for example).**

- **If the driving charge is slightly offset from the z-axis we expand the rhs of (1.80) retaining only the first order terms in the offset r_1:**

$$W_P(r,r_1,s) \sim W_P(r,0,s) + \left[\nabla_{\perp,r_1} W_P(r,r_1,s)\right]_{r_1=0}.r_1 + O(r_1^2) \qquad \textbf{(1.82)}$$

- **Thus (1.80) becomes:**

$$\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r,r_1,s) = -\nabla_{\perp,r}\left\{W_P(r,0,s) + \left[\nabla_{\perp,r_1} W_P(r,r_1,s)\right]_{r_1=0} . r_1\right\} \qquad \textbf{(1.83)}$$

- **The first term is in fact a monopole contribution to the transverse impedance and this often disappears according to the geometry (circular, rectangular elliptic). The remaining term is the dipole impedance:**

$$\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r,r_1,s) = -\nabla_{\perp,r}\left[\nabla_{\perp,r_1} W_P(r,r_1,s)\right]_{r_1=0} . r_1 \qquad \textbf{(1.84)}$$

Mode Expansion of Transverse Wake Function in Coordinates with Cylindrical Symmetry

- **As in the case of the longitudinal wake function, the transverse wake function is expressed as a superposition of multipole terms:**

$$w_{\perp}(r,r_l;\tau)=\sum_{m=0}^{\infty} w_{\perp,m}(r,r_l;\tau) \qquad \textbf{(1.85)}$$

- **Applying Panofsky-Wenzel(1.80) and making use of the expansions of the longitudinal wake function(1.32), (1.35) we obtain:**

$$\frac{\partial}{\partial s} w_{\perp,m}(r,r_l,s)=-\overline{\overline{w}}_{z,m}(s)r^{m-1}r_l^m\left\{\cos(m\phi)\hat{r}-\sin(m\phi)\hat{\phi}\right\} \qquad \textbf{(1.86)}$$

- **It is interesting to note that the dipole term, m=1 is linearly proportional to the offset of the driving charge and it is independent of the witness charge. The dipole transverse force is directed along the offset of the leading charge:**

$$\frac{\partial}{\partial s} w_{\perp,1}(r,r_l,s)=-\overline{\overline{w}}_{z,m}(s)r_l \qquad \textbf{(1.87)}$$

- **where $\overline{\overline{w}}_{z,m}(s)$ is the amplitude of the dipole longitudinal wake function (V/C/m^2)**

Transverse Coupling Impedance

- **The impedance is defined in terms of the Fourier transform of the transverse wake function with the additional imaginary factor (Ohms):**

$$Z_\perp(r,r_2;\omega) = j\int_{-\infty}^{\infty} w_{\perp,1}(r,r_2;\tau)\exp\left(-j\omega\tau\right)d\tau \qquad \textbf{(1.88)}$$

- **The imaginary constant was introduced in order to make the transverse impedance play the same role as the longitudinal one in beam stability theory.**
- **The dipole wake is usually the dominant one therefore it is natural to normalize with respect to the offset of the drive bunch (Ohms/m):**

$$Z'_\perp(r_1,r_2;\omega) = \frac{Z_\perp(r_1,r_2;\omega)}{r_1} \qquad \textbf{(1.89)}$$

The transverse wake is obtained via the inverse Fourier transform:

$$w_{\perp,1}(r,r_2;\tau) = -\frac{j}{2\pi}\int_{-\infty}^{\infty} Z_\perp(r,r_2;\omega)\exp\left(j\omega\tau\right)d\omega \qquad \textbf{(1.90)}$$

The Fourier transform of (1.80) gives the dipole transverse impedance in terms of the longitudinal one (Ohms):

$$\mathrm{FT}\left\{\frac{\partial}{\partial s}\mathbf{W}_\perp(r,r_1,s) = -\nabla_\perp W_P(r,r_1,s)\right\}$$

$$\Rightarrow Z_\perp(r,r_1;\omega) = -\frac{c}{\omega}\nabla_\perp Z(r,r_1;\omega) \qquad \textbf{(1.91)}$$

For an arbitrary shape (1.84) gives:

$$FT\left\{\frac{\partial}{\partial s}\mathbf{W}_{\perp}(r,r_1,s) = -\nabla_{\perp,r}\left[\nabla_{\perp,r_1}W_P(r,r_1,s)\right]_{r_1=0}.r_1\right\}$$

$$\Rightarrow Z_{\perp}(r,r_1;\omega) = \frac{c}{\omega}\nabla_{\perp,r}\left[\nabla_{\perp,r_1}Z(r,r_1,s)\right]_{r_1=0}.r_1 \qquad \textbf{(1.92)}$$

In cylindrical symmetry we obtain:

$$Z_{\perp,1}(r,r_1;\omega) = \frac{c}{\omega}\overline{\overline{Z_1}}(\omega)r_1 \qquad \textbf{(1.93)}$$

where (1.87) and (1.36) have been used.

Modal Sum Representation of Wakefield via Field Function Analysis

It will be shown that for any cavity the wakefield may be expanded in a modal sum:

$$W_p(s) = \sum_n 2\kappa_n \cos(\omega_n s/c) \ \forall s > 0 \qquad \textbf{(1.94)}$$

Where s>0 is refers to the distance behind the driving bunch (by causality for s<0 then W =0). The κ_n are the characteristic loss factors of the structure and ω_n are the cavity resonance frequencies; both of which are readily calculated with computer codes, HFSS, KN7C, Microwave Studio, MAFIA or OMEGA3 (a finite element computer code code developed at SLAC), for example.

In order to prove the above general expansion we resort to a vector and scalar potential representation of the fields:

$$\mathbf{E} = \frac{\partial}{\partial t}\mathbf{A} - \nabla\Phi, \ \ \mathbf{B} = \nabla \mathrm{x} \mathbf{A} \qquad \textbf{(1.95)}$$

Substituting these relations into Maxwell's equations readily yields:

$$\nabla^2 \mathbf{A} - c^{-2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{j} + c^{-2}\frac{\partial}{\partial t}\nabla\Phi \qquad \textbf{(1.96)}$$

$$\nabla^2 \Phi = -\varepsilon_0^{-1}\rho \qquad \textbf{(1.97)}$$

Where a Coulomb gauge has been used for the potentials:

$$\nabla.\mathbf{A}=0 \qquad \textbf{(1.98)}$$

The vector potential itself can be expanded into the modes a_n of the closed resonator structure:

$$\mathbf{A}(r,t)=\sum_{n} q_n(t)\mathbf{a}_n(r) \qquad \textbf{(1.99)}$$

where the a_n are required to satisfy the equation:

$$\left[\nabla^2+(\omega_n/c)^2\right]\mathbf{a}_n(r)=0 \qquad \textbf{(1.100)}$$

where the ω_n are the structure eigenfrequencies. Also, the a_n form a complete orthogonal set of basis vectors and we choose:

$$\frac{\varepsilon_0}{2}\int d^3r\,\mathbf{a}_\mathbf{n}^*(\mathbf{r}).\mathbf{a}_\mathbf{m}(\mathbf{r})=U_n\delta_{nm} \qquad \textbf{(1.101)}$$

where U_n is a normalizing factor and δ is the usual kronecker delta function.
This allows the wave equation to be rewritten in the form:

$$U_n\left[\omega_n^2+\frac{d^2}{dt^2}\right]q_n(t)=\frac{1}{2}\int d^3r\,\mathbf{a}_\mathbf{n}^*(r,t).\mathbf{j}(r,t)-\frac{\varepsilon_0}{2}\frac{\partial}{\partial t}\int d^3r\,\mathbf{a}_\mathbf{n}^*(r,t).\nabla\Phi(r,t) \qquad \textbf{(1.102)}$$

The second volume integral is transformed into a surface integral by Gauss' theorem and this vanishes

due to the boundary conditions imposed on Φ. Here, $\mathbf{a}_n^*.\nabla\Phi = \nabla(\mathbf{a}_n^*\Phi) - \Phi\nabla.\mathbf{a}_n^* = \nabla(\mathbf{a}_n^*\Phi)$ **has been used and the Coulomb gauge has been applied once more. Thus, the Fourier transform of the expansion coefficient is obtained as:**

$$\tilde{q}_n(\omega) = \int_{-\infty}^{\infty} dt\, q_n(t)\exp(-i\omega t) = \frac{1}{2U_n}\frac{1}{\omega-\omega_n^2}\int d^3r\, \mathbf{a}_n^*.\tilde{\mathbf{j}}(\mathbf{r},\omega) \qquad \textbf{(1.103)}$$

The Fourier transform of the vector potential is:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \sum_{\mathbf{n}} \tilde{q}_n(\omega)\mathbf{a}_n(\mathbf{r}) \qquad \textbf{(1.104)}$$

The scalar potential is also expanded into a complete orthonormal system:

$$\Phi(\mathbf{r},\omega) = \sum_{\mathbf{n}} r_n(\omega)\phi_n(\mathbf{r}) \qquad \textbf{(1.105)}$$

and ϕ satisfy the equation:

$$\left[\nabla^2 + (\Omega_n/c)^2\right]\phi_n(r) = 0 \qquad \textbf{(1.106)}$$

with boundary conditions that ϕ is zero on the surface surrounding the volume of the cavity. The orthogonalisation condition is chosen to define T such that:

$$\frac{\varepsilon_0}{2}\left(\Omega_n/c\right)^2\int d^3r\, \phi_n^*(r)\phi_m(r) = T_n\delta_{nm} \qquad \textbf{(1.107)}$$

This allows the expansion coefficients of Φ to be obtained as:

$$r_n(t) = \left(2T_n\right)^{-1} \int d^3r \phi_n^*(r)\rho(r,t) \quad \textbf{(1.108)}$$

Evaluation of Impedance:

The longitudinal impedance is defined as:

$$Z(x,y,s) = \frac{1}{q}\int_{-\infty}^{\infty} dz \tilde{E}_z(\mathbf{r},\omega)\exp(i\omega z/c) \quad \textbf{(1.109)}$$

The electric field $\tilde{\mathbf{E}}(\mathbf{r},\omega) = i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega) - \nabla\tilde{\Phi}(\mathbf{r},\omega)$ is excited by a charge q and here we consider a point charge moving parallel to the z-axis:

$$\begin{aligned} \rho(\mathbf{r},t) &= q\delta(z-ct)\delta(x-x_0)\delta(y-y_0) \\ \mathbf{j}(\mathbf{r},t) &= c\mathbf{e}_z\rho(\mathbf{r},t) \end{aligned} \quad \textbf{(1.110)}$$

The Fourier transform of the charge density and current are given by:

$$\begin{aligned} \tilde{\rho}(\mathbf{r},\omega) &= \frac{q}{c}\exp(-i\omega z/c)\delta(x-x_0)\delta(y-y_0) \\ \tilde{j}_z(\mathbf{r},\omega) &= q\exp(-i\omega z/c)\delta(x-x_0)\delta(y-y_0) \end{aligned} \quad \textbf{(1.111)}$$

Making use of the relations for r_n and q_n, derived above then the electric field is obtained as:

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},\omega) = q\sum_n \frac{-i\omega}{\omega^2-\omega_n^2}\mathbf{a}_n(\mathbf{r})\frac{1}{2U_n}\int_{-\infty}^{\infty} dz' a_{nz}^*(x_0,y_0,z')\exp(-i\omega z'/c) \\
-\frac{q}{c}\sum_n \nabla\phi_n^*(x_0,y_0,z')\exp(-i\omega z'/c)
\end{aligned}
\quad \textbf{(1.112)}
$$

The complex voltages are defined:

$$
\begin{aligned}
V_n(x,y,\omega) &= \int_{-\infty}^{\infty} dz\, a_{zn}(x,y,z)\exp(i\omega z/c) \\
v_n(x,y,\omega) &= \int_{-\infty}^{\infty} dz\left(\frac{\partial}{\partial z}\phi_n(x,y,z)\right)\exp(i\omega z/c)
\end{aligned}
\quad \textbf{(1.113)}
$$

Thus the longitudinal impedance is written as:

$$
\begin{aligned}
Z_0(x,y,\omega) = \sum_n \frac{-i\omega}{\omega^2-\omega_n^2}\frac{1}{2U_n}V_n(x,y,\omega)V_n^*(x_0,y_0,\omega) \\
+\sum_n \frac{i}{\omega}\frac{1}{2T_n}v_n(x,y,\omega)v_n^*(x_0,y_0,\omega)
\end{aligned}
\quad \textbf{(1.114)}
$$

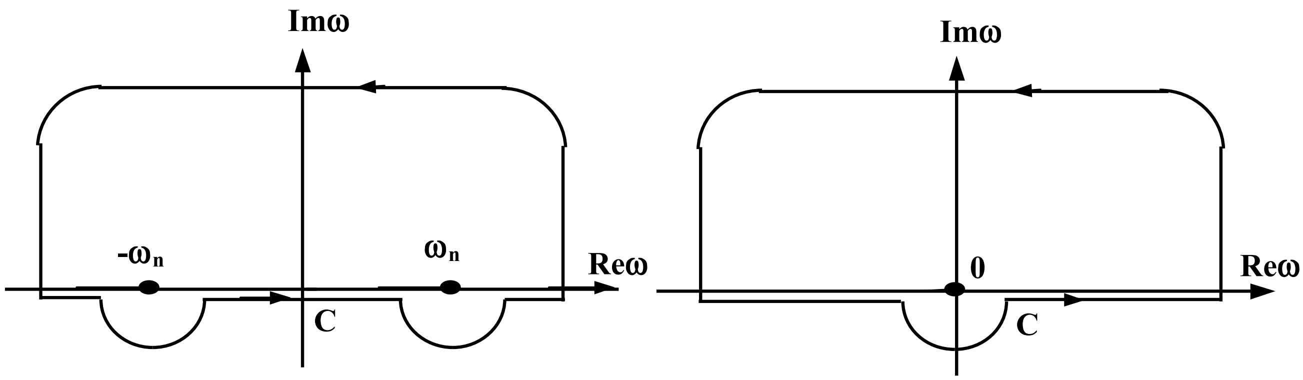
where the following relation, derived by integrating by parts, has been used:

$$
\frac{1}{c}\int_{-\infty}^{\infty} dz\,\phi_n^*(x_0,y_0,z)\exp(-i\omega z/c) = -\frac{i}{\omega}\int_{-\infty}^{\infty} dz\left(\frac{\partial}{\partial z}\phi(x_0,y_0,z)\right)\exp(-i\omega z/c) \quad \textbf{(1.115)}
$$

and the condition ϕ_n =0 at the boundary has been invoked.

The longitudinal Wake Potential is obtained by the inverse transform of the impedance:

$$W_p(x,y,s)=\frac{1}{2\pi}\oint_C d\omega Z(x,y,\omega)\exp(i\omega s/c) \qquad \textbf{(1.116)}$$



We integrate around the closed contour indicated and for s>0 we close the contour in the upper half plane and for s<0 we close the contour in the lower half plane. The second term of the impedance (which occurs due to the scalar potential) has a pole at ω=0 but it does not contribute to the wake because:

$$v_n(x,y,0)=\int_{-\infty}^{\infty}dz\left(\frac{\partial}{\partial z}\phi_n(x,y,z)\right)=\left[\phi_n(x,y,z)\right]_{\text{boundary}} \qquad \textbf{(1.117)}$$

Thus the wake function is given as:

$$\begin{aligned}W_p(x,y,s)=\sum_n\frac{1}{4U_n}&[V_n(x,y,\omega_n)V_n^*(x_0,y_0,\omega_n)\exp(i\omega_n s/c)\\&+V_n(x,y,-\omega_n)V_n^*(x_0,y_0,-\omega_n)\exp(-i\omega_n s/c)]\end{aligned} \qquad \textbf{(1.118)}$$

We are at liberty to choose real eigenvectors a_n and this makes:

$$V_n^*(x,y,-\omega) = V_n(x,y,\omega) \qquad \textbf{(1.119)}$$

and we will specialize to the case {x,y} = {x_0,y_0} to give:

$$W_P(x,y,s) = \sum_n 2\frac{|V_n(x,y,\omega_n)|^2}{4U_n}\cos(\omega_n s/c) \quad \forall s>0 \qquad \textbf{(1.120)}$$

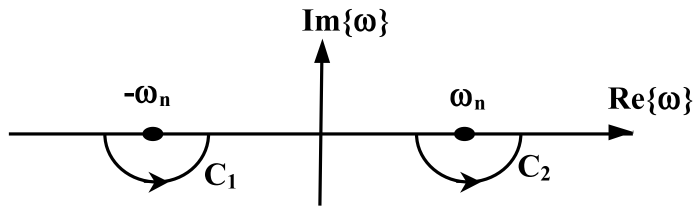
The loss parameter, of the transverse mode is given by:

$$\kappa_n = |V_n(x,y,\omega_n)|^2/(4U_n) \qquad \textbf{(1.121)}$$

It remains to calculate the wakefield at s = 0:

$$W_0(x,y,0) = \frac{1}{2\pi}\oint_C d\omega Z_0(x,y,\omega) \qquad \textbf{(1.122)}$$

The impedance function is an odd function ($Z_0(x,y,\omega) = -Z_0(x,y,-\omega)$) and thus it must be evaluated for the two contours C_1 and C_2 which consist of semi-circles with radius ε (where the limit ε->0 will be taken).



The contour C_1 gives:

$$\frac{1}{2\pi}\int_{C_1} d\omega \frac{-\omega}{\omega^2-\omega_n^2}\frac{|V_n(x,y,\omega)|^2}{2U_n} = \frac{1}{2}\frac{1}{i\pi}\int_{C_1} d\omega \frac{1}{\omega+\omega_n}\frac{|V_n(x,y,\omega)|^2}{2U_n}$$
$$= \frac{1}{2}\frac{1}{i\pi}\int_0^{\pi} d\phi \frac{i|V_n(x,y,\omega_n-\varepsilon e^{i\phi})|^2}{4U_n} \qquad \textbf{(1.123)}$$
$$= \frac{1}{2}\kappa_n$$

Similarly for the contour C_2 and thus in general the longitudinal wakefield is given by:

$$W_p(s) = \theta(s)\sum_n 2\kappa_n \cos(\omega_n s/c) \qquad \textbf{(1.124)}$$
$$\theta(s) = \begin{cases} 0 & s<0 \\ 1/2 & s=0 \\ 1 & s>0 \end{cases}$$

Causality is expressed in this equation by the fact that the wakefield is zero ahead of the driving bunch. Further, what is sometimes called the *fundamental theorem of beam loading* is expressed by the factor of ½ which describes the wakefield felt by the driving bunch itself.

Practical Wake Function Expansions

The longitudinal and transverse wake functions are written:

$$W_p(\mathbf{r_1},\mathbf{r};s) = 2\theta(s)\sum_n \kappa_n(\mathbf{r_1},\mathbf{r})\cos(\omega_n s/c) \qquad \textbf{(1.125)}$$

$$\mathbf{W}_\perp(\mathbf{r_1},\mathbf{r};s) = 2\theta(s)\sum_n \boldsymbol{\kappa}_{n\perp}(\mathbf{r_1},\mathbf{r})\sin(\omega_n s/c) \qquad \textbf{(1.126)}$$

where the longitudinal and transverse loss factors are given by:

$$\kappa_n = \frac{V_n^*(\mathbf{r}_1)V_n(\mathbf{r})}{4U_n},\ \boldsymbol{\kappa}_{n\perp} = \frac{V_n^*(\mathbf{r}_1)\nabla_\perp V_n(\mathbf{r})}{4U_n\omega_n/c} \qquad \textbf{(1.127)}$$

and U_n is the energy stored in a particular mode n and the voltage evaluated from the integral of the axial electric field along L, the length of the cavity:

$$V_n(\mathbf{r}) = \int_0^L E_{zn}(r,z)\exp\left(\frac{i\omega_n z}{c}\right)dz \qquad \textbf{(1.128)}$$

and a similar expression for $V_n(\mathbf{r_1})$.

The transverse wake function is zero at the bunch (s=0) and it increases linearly in close proximity behind the bunch. There is no wake in front of the bunch from causality considerations.

These wake functions are valid for v=c. For v<c additional correction terms of $O(\gamma^{-2})$ occur.

The m^th order multipole wake functions are:

$$W_P^m(s) = 2\theta(s)\left(\frac{r_1}{a}\right)^m\left(\frac{r}{a}\right)^m \sum_n \kappa_n^m(\mathbf{r_1},\mathbf{r})\cos(\omega_n^m s/c) \qquad \textbf{(1.129)}$$

$$\mathbf{W}_\perp^m(s) = 2\theta(s)\left(\frac{r_1}{a}\right)^m\left(\frac{r}{a}\right)^{m-1}\left[\hat{r}\cos m\theta - \hat{\theta}\sin m\theta\right]\sum_n \frac{\boldsymbol{\kappa}_n}{\omega_n^m a/c}(\mathbf{r_1},\mathbf{r})\sin(\omega_n^m s/c) \qquad \textbf{(1.130)}$$

In particular for calculations on X-band structures for the NLC, the dipole (m=1) wake function is often computed in the form:

$$\mathbf{W}_d(s) = \frac{W_\perp(s)}{r_1 L} = 2\theta(s)\sum_n K_n(\mathbf{r_1})\sin(\omega_n^m s/c) \qquad \textbf{(1.131)}$$

where the "kick factor" is defined in terms of the loss factor:

$$K_n = \frac{c\boldsymbol{\kappa}_n}{\omega_n a^2 L} \qquad \textbf{(1.132)}$$

Both the kick factor and specially defined dipole wake have units of V/C/m^2, and for X-band high energy linacs these units are often rewritten as: V/pC/mm/m.

Why use these strange units?

- **The millimeter factor arises from the offset of the drive bunch and this is usually of the order of mm (the iris has a radius of ~4mm or so).**
- **The per meter factor arises because each accelerating structure is of the order of 1m or less.**
- **The beam usually has a charge of ~ a few pC (approx 1.1 10^{10} particles are present in one bunch).**

To obtain the transverse momentum change of a trailing particle due to the wake left behind the drive particle:

- **multiply the specially defined dipole kick factor by the driving and witness charge, the length of the accelerating structure, and the offset of the drive charge and divide by the velocity of light ($\Delta p = Kqq_l La/c$).**
- **In terms of the normal kick factor one would compute $\Delta p = Kqq_l/c$.**

Wake Function in A Pill Box Cavity

- **A closed off, circular cylinder is taken as the cavity to analyze.**
- **In microwave parlance this is often known as a "pill box" cavity.**
- **This cavity permits a formally exact calculation of the wake functions. However, few of the summations can be evaluated in closed form.**

  - **Before proceeding, it is worth noting the general feature that the summation of the series of modes which describe the wake function converges rather slowly within the bunch itself.**
  - **Nonetheless, for positions well behind the bunch, the series converges much faster and hence the energy loss can be accurately evaluated.**

To evaluate the wake function we will use:

$$G(s) = 2\sum_{\mu} k_{\mu} \cos\left(\omega_{\mu}\frac{s}{c}\right), \quad k_{\mu} = \frac{V_{\mu}V_{\mu}}{4U_{\mu}} \qquad \textbf{(1.133)}$$

Multiplying by the charge of the drive particle gives in the potential seen by the trailing particle.

The longitudinal modes in the cavity are given by:

$$\omega_{np}^2 / c^2 = \left(j_n / R \right)^2 + \left(\pi p / g \right)^2 = \nu_{np}^2 \qquad \textbf{(1.134)}$$

where is the nth solution of $J_0(j_n)=0$, g is the length of the cavity and R is the radius. The cavity fields are:

$$\begin{aligned} E_z^{n,p} &= \frac{j_n}{R} J_0\left(j_n \frac{r}{R} \right) \cos\left(\frac{\pi p z}{g} \right) \exp\left(i\omega_{np} t \right) \\ E_r^{n,p} &= \frac{\pi p}{g} J_1\left(j_n \frac{r}{R} \right) \sin\left(\frac{\pi p z}{g} \right) \exp\left(i\omega_{np} t \right) \\ H_\theta^{n,p} &= i\omega_{np}\varepsilon_0 J_1\left(j_n \frac{r}{R} \right) \cos\left(\frac{\pi p z}{g} \right) \exp\left(i\omega_{np} t \right) \end{aligned} \qquad \textbf{(1.135)}$$

The voltage is evaluated on the axis (r=0):

$$V_{np} = \int_0^g E_z(r=0, z, t=z/c) = \frac{i\nu_{np} R}{j_n}\left[1 - (-1)^p \exp\left(i\nu_{np} g \right) \right] \qquad \textbf{(1.136)}$$

and thus:

$$V_{np} V_{np}^* = 2\left(\frac{\nu_{np} R}{j_n} \right)^2 \left[1 - (-1)^p \cos\left(\nu_{np} g \right) \right] \qquad \textbf{(1.137)}$$

The energy stored in the cavity is given by:

$$U_{np} = \int_0^R dr \int_0^{2\pi} d\theta \int_0^g dz\, H_\theta^{np} H_\theta^{*np} = \frac{\pi\varepsilon_0}{4} \nu_{np}^2 g R^2 J_1^2(j_n) \qquad \textbf{(1.138)}$$

Thus, the loss factor is evaluated as:

$$k_{np} = \frac{1}{\pi\varepsilon_0 g}\frac{2}{1+\delta_p^0}\frac{1-(-1)^p\cos(\nu_{np}g)}{j_n^2 J_1^2(j_n)} \qquad \textbf{(1.139)}$$

where δ_p^0 is the Kronecker delta function. The point charge longitudinal wake function is then given by the double summation:

$$\pi\varepsilon_0 g G(s) = 2\theta(s)\sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty}\frac{1-(-1)^p\cos(\nu_{np}g)}{j_n^2 J_1^2(j_n)}\cos(\nu_{np}s) \qquad \textbf{(1.140)}$$

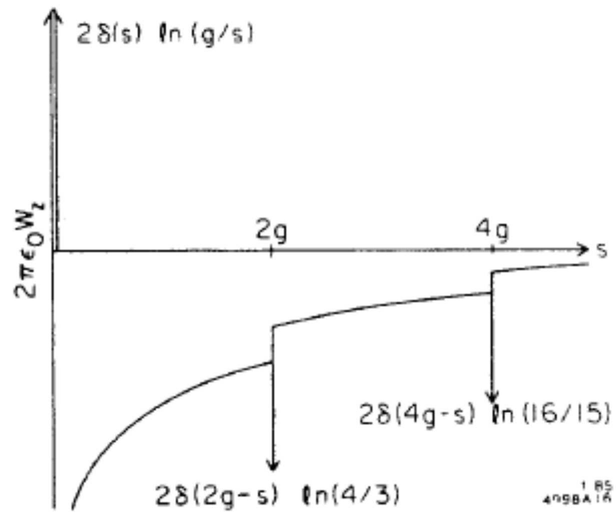
It is not possible, in general, to obtain a closed from to the above sum. However, for the special case of $s < s_0 \equiv \sqrt{4R^2+g^2}-g$ the sums can partially be evaluated to a series of delta functions:

$$2\pi\varepsilon_0 g G(s) = 2\delta(s)\ln\left[\frac{g}{s}\right] -$$

$$2\sum_{n=1}^{\infty}\delta(2ng-s)\ln\left[\frac{s^2}{s^2-g^2}\right] - \frac{1}{g}\left\{\left(\left[\frac{s}{2g}\right]_{IP}+\frac{s}{2g}\right)^{-1} - \left(\left[\frac{s}{2g}\right]_{IP}+\frac{s}{2g}+1\right)^{-1}\right\} \qquad \textbf{(1.141)}$$

- **Applying causality, this wake must be the same as that produced in between two parallel plates. That is, no signal is able to propagate from the point where the driving charge enters the cavity, be reflected from the outer wall, and return to the path followed by the driving charge with a distance s_0 behind it.**

• **The wake is accelerating at all points accept at s = 0. The driving charge itself experiences an infinite retarding potential at the moment it exits the second plane of two parallel plates.. Spherical wavefronts, which expand with the velocity of light, are generated when the charge enters through the first plane and again when it leaves through the second plane. On the axis two of these wavefronts join in the double cusp geometry shown at position X. When a trailing particle meets and passes through this singularity or a later reflection of it, it will experience a finite accelerating potential given by the third term on (1.141). For small s this accelerating potential diverges as 1 /s. If s is a multiple of 2g the test particle will travel with the singularity across the cavity and experience an infinite accelerating potential.**



The convolution of the point source wake in (1.140) with a Gaussian current:

$$I(s)=\frac{\exp(-\frac{s^2}{2\sigma^2})}{\sqrt{2\pi}\sigma} \qquad \textbf{(1.142)}$$

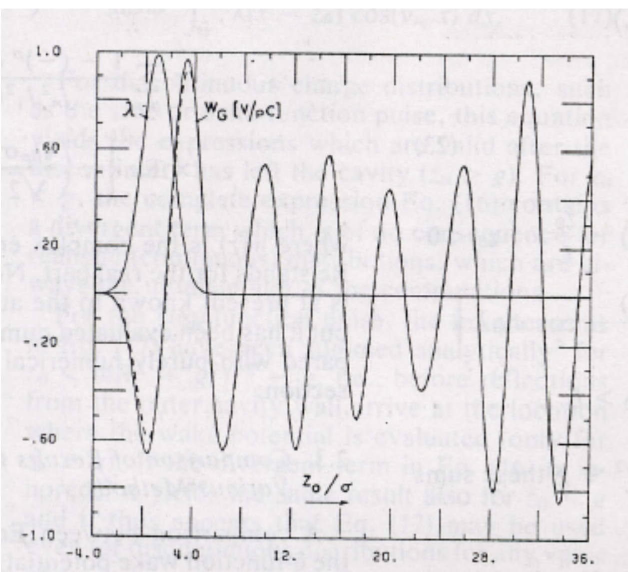
gives the Gaussian bunch wake function:

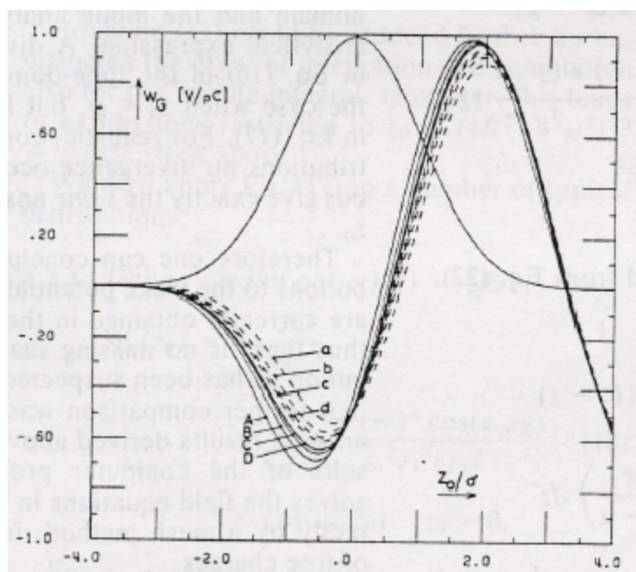
$$\pi\varepsilon_0 g G(s)=$$

$$\theta(s)\exp\left(-\frac{s^2}{2\sigma^2}\right)\sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty}\frac{1-(-1)^p\cos\left(\nu_{np}g\right)}{j_n^2 J_1^2(j_n)}\mathrm{Re}\left\{w\left(\frac{\nu_{np}\sigma}{\sqrt{2}}-\frac{is}{\sqrt{2}}\right)\right\} \qquad \textbf{(1.143)}$$

where w(z) is the complex error function:

$$\int_0^{\infty}dt\exp\left(-a^2t^2+izt\right)=\frac{\sqrt{\pi}}{2a}w\left(\frac{z}{2a}\right) \qquad \textbf{(1.144)}$$





Computed Pill-Box Wake Function for Gaussian Bunch

Mode results are shown with solid lines. Dashed lines correspond to a time domain simulation. The fig on the right is shows the effect of changing the number of modes: a,b,c,d =10, 40, 160, and 64 modes, respectively.

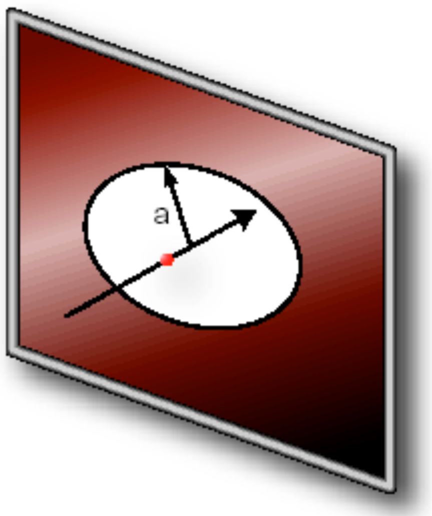
- **"Outside" the realm of the bunch, i.e. $|s|>4\sigma$ the wake function is accurately computed with a limited number of modes –the time domain and modal analysis agree very well! Ten modes is sufficient in this particular example for an accurate computation of the wake.**

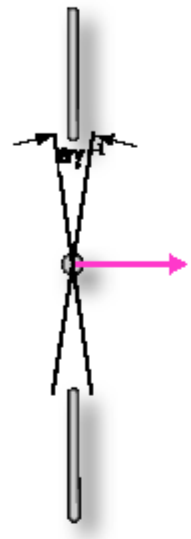
- **Inside the region of the bunch, i.e. $|s|<4\sigma$ many modes are necessary in order to accurately compute the wake function. As the number of modes is increased and the number of mesh points in the time domain method is increases both methods converge towards the same value -they converge from opposite ends.**

- **The wake function is decelerating within the bunch. Is this physically correct? Yes, otherwise the bunch would be continuously accelerated by its own field. Although, the tail of the bunch is in an accelerating region and this does permit acceleration of the tail by the head of the bunch.**

- **There is no damping in this system. Hence the oscillation in the wake is allowed to rise up again at some point.**

Point Charge Passing through an Iris





- **Iris cuts off part of the electromagnetic that hits the metal.**
- **Duration of the field pulse on iris** $\sim \Delta t = a/c\gamma$
- **Let us calculate the energy U of the e.m. field that is clipped off by the iris.**
- **The fields of the ultra relativistic charge are:**

$$E_r = \frac{\gamma q r}{4\pi\varepsilon_0 (r^2 + \gamma^2 z^2)^{3/2}} \qquad \textbf{(1.145)}$$

➢ **The energy density:**

$$w = \frac{1}{2}\varepsilon_0 E_r^2 + \frac{1}{2}\mu_0 H_\phi^2 = \varepsilon_0 E_r^2 \qquad \textbf{(1.146)}$$

➢ **Integrating w over the region r > a (the region clipped by the iris) and over z:**

$$U = \int_a^\infty 2\pi r dr \int_{-\infty}^{\infty} w dz = \frac{3q^2\gamma}{4^4 \varepsilon_0 \pi a^2} \qquad \textbf{(1.147)}$$

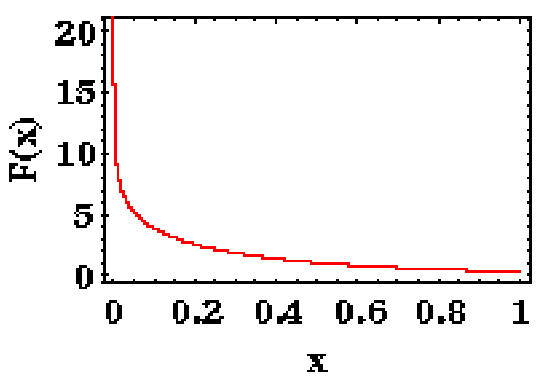
➢ **We expect the radiated energy is of the order of U. The spectrum of radiation involves the frequencies up to** $D \sim a/\gamma$ **(**$D=1/k$**).**

➢ **The problem allows an analytical solution using a diffraction model. The energy density with respect to frequency is found for k>>1/a:**

$$P(\omega) = \frac{q^2}{8\pi^3 \varepsilon_0 c} F\left(\frac{ak}{\gamma}\right) \qquad \textbf{(1.148)}$$

where:

$$F(x) = x^2\left[K_0(x)K_2(x) - K_1^2(x)\right] \textbf{(1.149)}$$



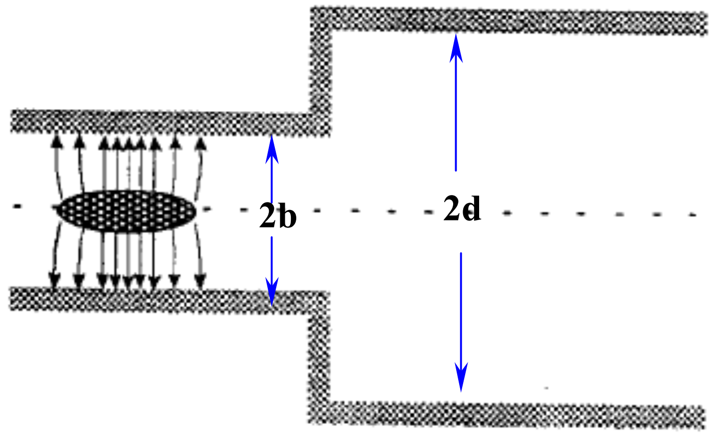
with K_n the modified Bessel functions of the second kind.

- **Integrating the energy density with respect to frequency:**

$$\int_0^\infty d\omega P(\omega) = \frac{3q^2\gamma}{128\varepsilon_0 \pi a^2} \qquad \textbf{(1.150)}$$

- **Thus, the radiated energy is twice the clipped energy. This is because the clipped field is reflected back by the screen and is radiated in the backward direction. The same amount of energy is radiated in the forward direction when the charge gives rise to the new field**

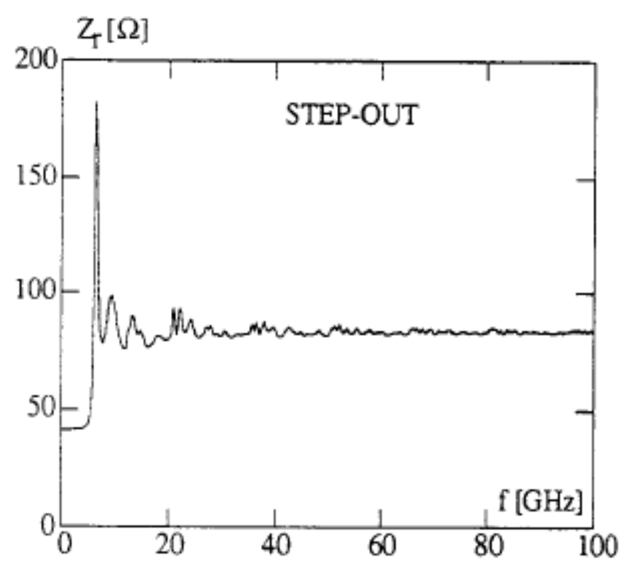
Step-Out Transitions in Waveguide



- We consider a transition in which the radius increases from b to d
- Beam is shown entering from the left
  - The contribution to impedance is mainly resistive for this case
- When the charge crosses the boundary discontinuity the self-field restoring the boundary conditions has to fill the extra space b<r<d between the two pipes, whilst diffracted fields propagate into the pipes.
- Energy loss is given by:

$$q^2 k^{out} = U(b<r<d) + E_{rad} \qquad \textbf{(1.151)}$$

where E_{rad} is the energy radiated at the edges and U(b<r<d) is the energy necessary to fill the region b<r<d.



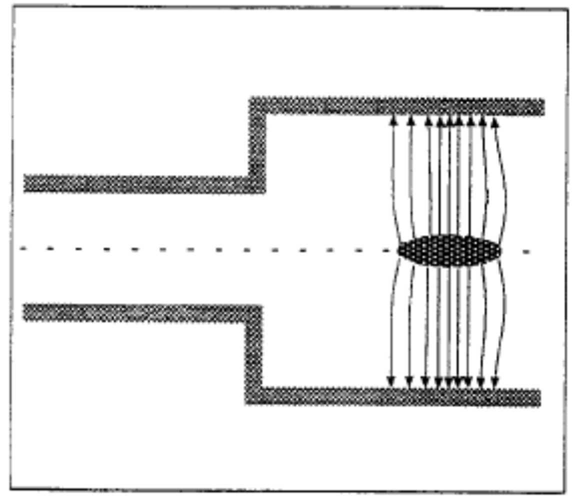
The asymptotic(high frequency) behavior of the real part of the impedance is:

$$Z_{m=0}^{out} = \frac{Z_0}{\pi} \ln\left(\frac{d}{b}\right) \qquad \textbf{(1.152)}$$

➢ This result can be derived from energy considerations

For an ultra relativistic beam the radial electric field in the tube is given by:

$$E_r \sim \frac{q}{2\pi\varepsilon_0 r}\left(\frac{\gamma}{b}\right) \qquad \textbf{(1.153)}$$



The energy is given by:

$$U(b<r<d) = \int_V E_r^2 dV \qquad \textbf{(1.154)}$$

and this makes the loss factor:

$$k^{out} = 2\frac{U}{q^2} = \frac{Z_0}{\pi}\ln\left(\frac{b}{d}\right)\left(\frac{c\gamma}{b}\right) \qquad \textbf{(1.155)}$$

The effective charge size is $\sigma_{eff} = b/\gamma$ and the loss factor is inversely proportional to this size, and thus we expect $Z_r(\omega)$ is a constant function of frequency. Hence, the overall loss factor is given by:

$$K = \frac{1}{\pi}\int_0^\infty Z_r(\omega)\frac{\sin^2(\omega\sigma_{eff}/2c)}{(\omega\sigma_{eff}/2c)^2}d\omega = \frac{Z_r(\omega)}{\sigma_{eff}/c} \qquad \textbf{(1.156)}$$

Thus makes:

$$Z_r(\omega) = \frac{Z_0}{\pi}\ln\left(\frac{d}{b}\right) \qquad \textbf{(1.157)}$$

and we obtain the impedance as stated in (1.152).

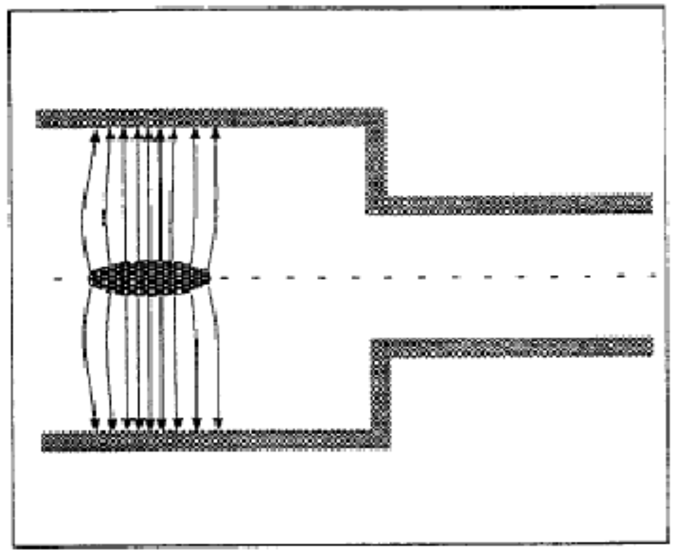
n.b. In in (1.156) we have used the Fourier transform, H, of the unit step function :
$FT\left\{\frac{1}{\sigma}H[\frac{s}{\sigma}]\right\} = \frac{1}{\sigma}\int_{-\infty}^{\infty} dt H[\frac{s}{\sigma}]\exp(-i\omega s/c) = \frac{\sin(\omega\sigma/2c)}{\omega\sigma/2c}$ and . $\int_0^\infty dx \frac{\sin ax}{ax} = \frac{\pi}{2a}$.

n.b. 2 The derivation is not rigorous but illustrative of the physics.

Step-In Transitions in Waveguide

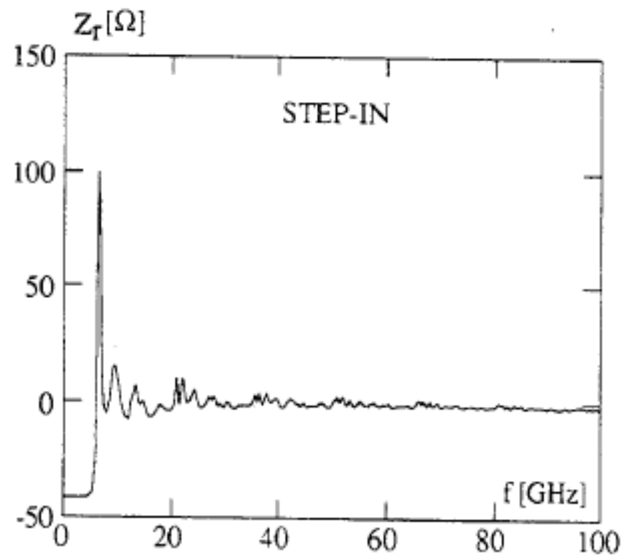
The physics of the step-in transition are quite different. The asymptotic loss factor is now:

$$k_{m=0}^{in} = 0 \qquad \textbf{(1.158)}$$



This is explained in terms of the radiated energy being reflected back with respect to the particle motion without changing its energy:

$$q^2 k_{m=0}^{in} = -U(b<r<d) + E_{rad} \qquad \textbf{(1.159)}$$



For a point charge, the radiated energy is taken out of the energy missing in the smaller radius pipe: $E_{rad} = U(b<r<d)$

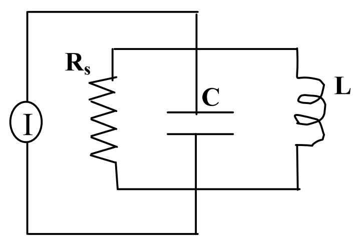
And thus:

$$q^2 k_{m=0}^{in} = 0 \qquad \textbf{(1.160)}$$

For a realistic bunch both U(b<r<d) and E_{rad} depend on the bunch length. Also, if the Fourier spectrum of the bunch does not appreciably cover the region of the pipe above cut-off then there is no radiation as the waveguide modes are below cut-off (all modes are evanescent).

R-L-C Circuit Model of Single Mode and Impedance-Wake Relations

Here, we point out some of the essential properties of impedance and this will serve as an introduction to the coupled two-band (TE-TM) circuit, coupled to a transmission line (representing a waveguide-like manifold) that will in forthcoming lectures, be used to model the wakefield in a DT (Detuned Structure) and in a DDS (Damped and Detuned Structure). However, here the basic features of R-L-C impedances and wakefields are delineated.



Each cell of the accelerating structure is represented by an R-L-C circuit. The circuit has a shunt impedance R_s, an inductance L and a capacitance C. In practice this represents the fields present in the structure and they cannot readily be measured. However, related quantities can be measured for a so simple R-L-C circuit, namely, the cavity resonance frequency, ω_r, the quality factor Q and the damping factor α:

$$\omega_r = (LC)^{-1/2}, \;\; Q = R_s (C/L)^{1/2}, \;\; \alpha = \omega_r / (2Q) \quad \textbf{(1.161)}$$

The circuit is driven by a current I and the voltages across each element are identical:

$$V = I_R R_s = \frac{1}{C}\int dt\, I_c = L\frac{dI_L}{dt} \quad \textbf{(1.162)}$$

Differentiating with respect to time t give the total current as:

$$\frac{dI}{dt} = \left(\frac{1}{R_s}\frac{d}{dt} + C\frac{d^2}{dt^2} + \frac{1}{L}\right)V \quad \textbf{(1.163)}$$

and this is readily rewritten as:

$$\left(\frac{d^2}{dt^2} + \frac{\omega_r}{Q}\frac{d}{dt} + \omega_r^2\right)V = \frac{\omega_r R}{Q}\frac{dI}{dt} \quad \textbf{(1.164)}$$

The solution is a damped oscillation:

$$V(t) = e^{-\alpha t} A\cos\omega_r' t + e^{-\alpha t} B\sin\omega_r' t,\ \ \omega_r' = \omega_r\left(1-(4Q^2)^{-1}\right)^{1/2} \quad \textbf{(1.165)}$$

The Wake Potential is calculated by enforcing a delta function driving current:

$$I(t) = q\delta(t) \quad \textbf{(1.166)}$$

This instantaneously induces a voltage across the capacitor:

$$V(0^+) = q/C = (\omega_r R_s/Q)q \quad \textbf{(1.167)}$$

The energy stored in the capacitor is equal to the energy lost by the point charge:

$$U = \frac{q^2}{C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = \kappa q^2 \qquad \textbf{(1.168)}$$

where κ is the mode loss factor:

$$\kappa = \frac{U}{q^2} = \frac{\omega_r R_s}{2Q} \qquad \textbf{(1.169)}$$

This capacitor then discharges through the resistor and through the inductance:

$$\left.\frac{dV}{dt}\right|_{t=0^+} = -\frac{1}{C}\frac{dq}{dt} = -\frac{I_R}{C} = -\frac{1}{C}\frac{V(0^+)}{R_s} = -\frac{\omega_r^2 R_s}{Q^2} q = -\frac{2\omega_r \kappa}{Q} q \qquad \textbf{(1.170)}$$

Thus we have the initial conditions:

$$V(0^+) = 2\kappa q \ \ \textbf{and} \ \ \dot{V}(0^+) = -\frac{2\omega_r \kappa}{Q} q \qquad \textbf{(1.171)}$$

Thus we now enforce these initial conditions in order to solve for the constants A and B. The differential of the voltage is given by:

$$\frac{dV}{dt} = e^{-\alpha t}[(-A\alpha + B\omega_r')\cos\omega_r' t - (B\alpha + A\omega_r')\sin\omega_r' t] \qquad \textbf{(1.172)}$$

Thus we obtain:

$$A = 2\kappa q \ \ \textbf{and} \ \ -A\alpha + B\omega_r' = -\frac{2\omega_r \kappa q}{Q} \qquad \textbf{(1.173)}$$

and this allows the voltage response to a delta function current excitation to be obtained as:

$$V(t) = 2q\kappa e^{-\alpha t}\left[\cos\omega_r' t - \frac{\sin\omega_r' t}{2Q(\omega_r'/\omega_r)}\right], \quad \omega_r' = \omega_r(1-(2Q)^{-2})^{1/2} \quad \textbf{(1.174)}$$

This voltage is induced by a point charge going through a cavity at t =0. A second charge q' will at a time t gain or loose energy U = q'V(t). This energy loss or gain per unit source and probe charge is the wake or Green function G(t). For this cavity resonance we have:

$$G(t) = 2\kappa e^{-\alpha t}\left[\cos\omega_r' t - \frac{\sin\omega_r' t}{2Q(\omega_r'/\omega_r)}\right] \quad \textbf{(1.175)}$$

Typically the quality factor is very high and thus:

$$G(t) = 2\kappa e^{-\alpha t}\cos\omega_r' t \quad \textbf{(1.176)}$$

as is consistent with the result derived previously using the field function analysis.

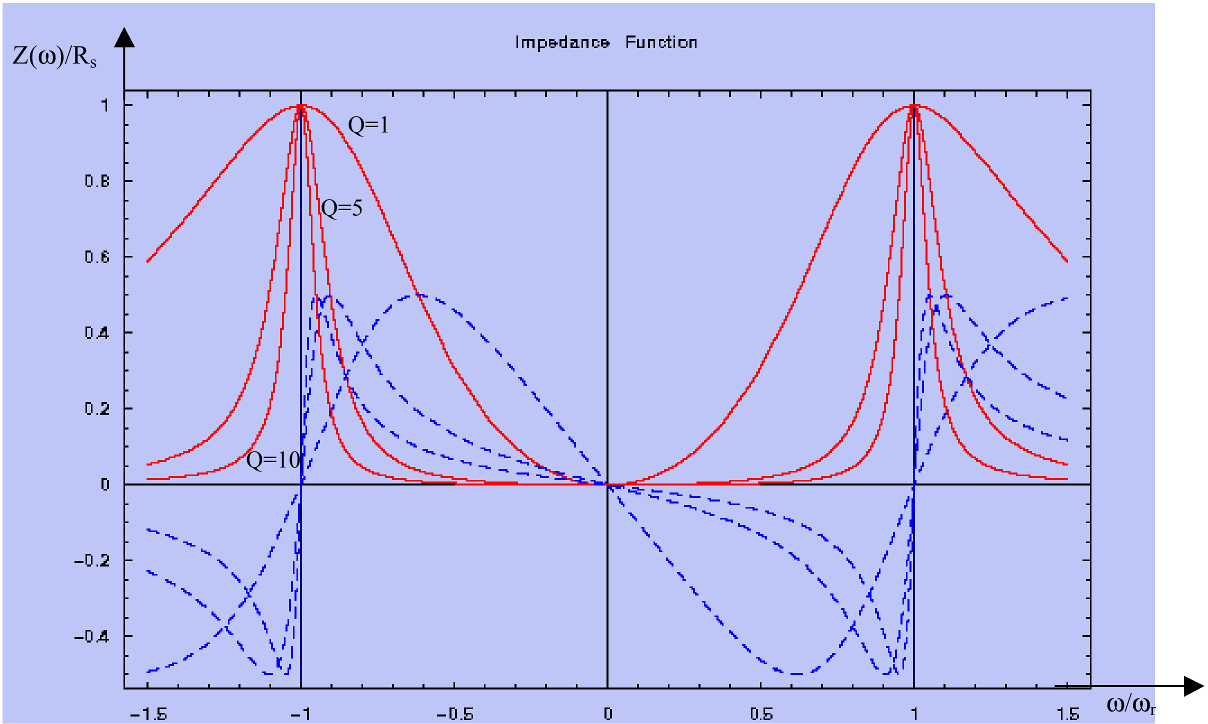
To evaluate the impedance we switch to a complex phasor notation:

$$I(t) = \hat{I}\exp(j\omega t), \quad V(t) = V_0\exp(j\omega t) \quad \textbf{(1.177)}$$

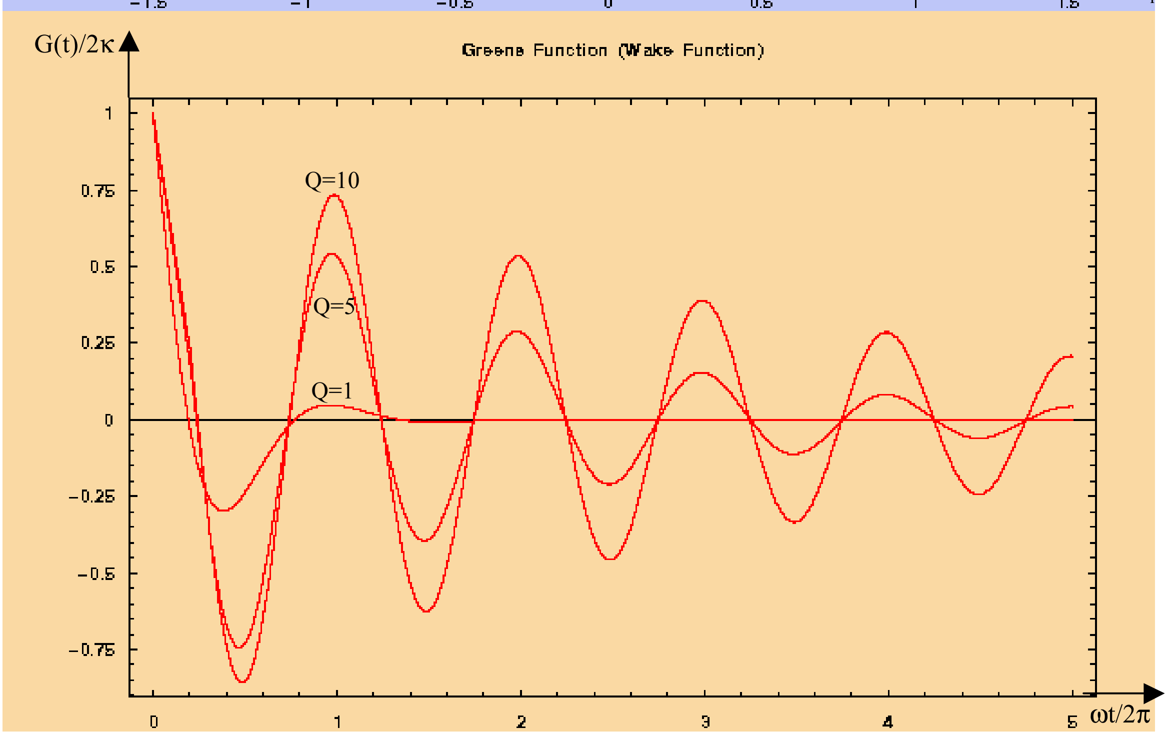
and thus the differential equation for the R-L-C circuit becomes:

$$\left(-\omega^2 + j\frac{\omega_r\omega}{Q} + \omega_r^2\right)V_0\exp(j\omega t) = j\frac{\omega_r\omega R_s}{Q}\hat{I}\exp(j\omega t) \quad \textbf{(1.178)}$$

The impedance is the ratio of the voltage to the current:



Z(ω)/Rs
Impedance Function
Q=1
Q=5
Q=10
ω/ωr



G(t)/2κ
Greens Function (Wake Function)
Q=10
Q=5
Q=1
ωt/2π

$$Z(\omega)=V/I=V_0/\hat{I}=R_s\frac{j\omega_r\omega Q^{-1}}{\omega_r^2-\omega^2+j\omega_r\omega Q^{-1}}=R_s\frac{1-jQ\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)}{1+Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2}=\text{Re}\{Z\}+j\text{Im}\{Z\}$$

(1.179)

For relatively loss-less systems with very high quality factors then the impedance is large in the vicinity of $\omega\approx\omega_r$ **or** $\Delta\omega/\omega_r(=|\omega-\omega_r|/\omega_r)=1$**and this allows the impedance to be simplified to:**

$$Z(\omega)=R_s\frac{1-j2Q\Delta\omega/\omega_r}{1+4Q^2(\Delta\omega/\omega_r)^2} \quad \textbf{(1.180)}$$

Resonator has the following properties, which are used in the coupled circuit design described in the following lectures:

$\omega=\omega_r\rightarrow Z_r(\omega_r)$ **has a maximum, and** $Z(\omega_r)=0$

$|\omega|<\omega_r\rightarrow Z_i(\omega)>0$ **the impedance is inductive**

$|\omega|>\omega_r\rightarrow Z_i(\omega)<0$ **the impedance is capacitive**

Further, for any impedance or potential it can readily be shown that:

$$Z_r(\omega)=Z_r(-\omega),\;\; Z_i(\omega)=-Z_i(\omega) \quad \textbf{(1.181)}$$

$Z(\omega)=\int_{-\infty}^{\infty}G(t)\exp(-j\omega t)dt$, **Z(ω) = the Fourier transform of G(t), the Wake function**

Methods of Wakefield Calculation

1. ***Finite difference + finite element codes => MAFIA, Omega3 (3-d frequency domain), Tau3(3-d time domain) GdfidL (3-D freq/time domain), ABCI (time domain), HFSS (finite difference freq. domain)***

2. Mode-matching (frequency domain) => Smart2D(2-d, match modes transversely), Transvrs (single periodic iris, match modes longitudinally), Cascade(2-d, match modes transversely)

3. Circuit models: Single and dual mode manifold-damped, frequency domain models

END OF LECTURE 2

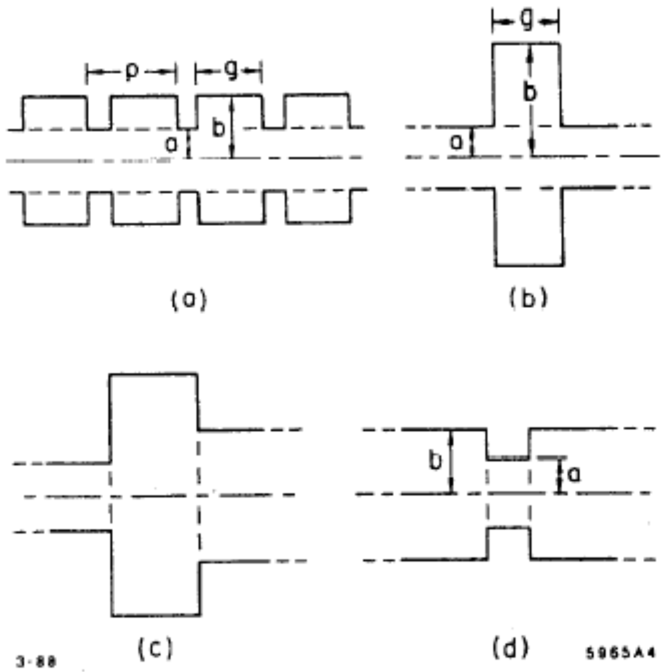
Mode-Matching Method

Well-established methods and computer codes have been developed to obtain the parameters of RF cavities, periodical structures and obstacles in a beam pipe. These can be divided into three main approaches: grid and finite difference/element based methods, secondly, frequency domain mode-matching methods, and thirdly, equivalent circuit based methods. Calculation of complex and 3D structures by the grid based methods is expensive computationally in terms of both memory and time. The mode-matching method has several advantages that make it useful for the simulations. The method permits simulation of complex structures in an efficient way and allows for qualitative analysis of the results which in a straightforward

manner. Mode matching is rather fast compared to grid based methods!

In constructing the field in the cavity structure the modes can be matched either longitudinally (as is done in the computer code *Transvrs*) or transverse to the direction of propagation (*Cascade* and *Smart2D* utilized this method). Matching the fields transversely readily allows the accelerator components to be stacked up or cascaded and allows for variations in the geometry of the components so that the overall field can be obtained.

Scattering Matrix Representation



Examples of Cylindrically Symmetric Accelerator Components with Jumps in Cross Sections

In the scattering formulation of a field problem, amplitudes of the scattered waves at each port are linear combinations of the amplitudes of the incident waves on each port.

Therefore, the scattering matrix S is an N x N matrix that relates the reflected waves at each port to the incident waves at each port. In a two-port network, N = 2, and the amplitudes of the scattered waves b_1, b_2 are related to the amplitudes of the incident waves a_1, a_2 (Fig. 1) as follows:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \qquad \textbf{(1.182)}$$

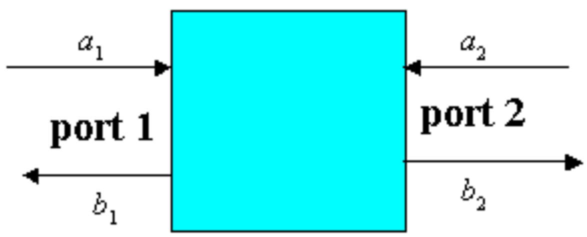


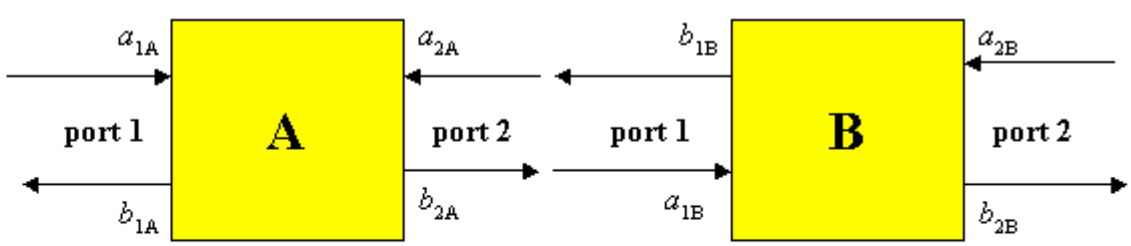
Figure 1: Two-port microwave network.

S_{11} is the reflection coefficient at the first port and characterizes how much of the incident wave a_1 is reflected at that port. S_{22} is the reflection coefficient at the second port, and S_{12}, S_{21} are the transmission

coefficients from port 2 to port 1 and vice versa. S_{12} characterizes how much of the incident wave at port 2 is transmitted through to port 1. More generally, in an N x N matrix, S_{ij} is the transmission coefficient from the j^{th} port to the i^{th} port, when every other port is terminated in a matched load.

Cascading of Two-Port Systems

Microwave networks can often be expressed as a cascade combination of two-port devices. Let the components A and B with S-matrices S^A and S^B respectively be connected sequentially as shown below. A simple, but extremely useful example of blocks A and B would be a change in waveguide radius from a to b, corresponding to a wide to a narrow transition for example. To construct an iris in waveguide would, in principle, require two cascades (although for a symmetrical waveguide-iris-waveguide transition only one cascade is required).



Two components (A and B) connected in a cascade.

If a_{1A}, b_{1A}, a_{1B}, etc represent normalized wave variables at the ports of the two components, then

$$\left.\begin{aligned} b_{1A} &= S^{A}_{11} a_{1A} + S^{A}_{12} a_{2A} \\ b_{2A} &= S^{A}_{21} a_{1A} + S^{A}_{22} a_{2A} \\ b_{1B} &= S^{B}_{11} a_{1B} + S^{B}_{12} a_{2B} \\ b_{2B} &= S^{B}_{21} a_{1B} + S^{B}_{22} a_{2B} \end{aligned}\right\} \qquad (1.183)$$

Port 2 of component A is connected to port 1 of component B (Fig.2):

$$b_{2A} = a_{1B} \quad \text{and} \quad b_{1B} = a_{2A}.$$

Thus, eliminating b_{2A}, b_{1B}, a_{1B} and a_{2A} from (2.1):

$$\begin{bmatrix} b_{1A} \\ b_{2B} \end{bmatrix} = \begin{bmatrix} S^{A}_{11} + S^{A}_{12} S^{B}_{11} S^{A}_{21} \left(1 - S^{A}_{22} S^{B}_{11}\right)^{-1} & S^{A}_{12} S^{B}_{12} \left(1 - S^{A}_{22} S^{B}_{11}\right)^{-1} \\ S^{A}_{21} S^{B}_{21} \left(1 - S^{A}_{22} S^{B}_{11}\right)^{-1} & S^{B}_{22} + S^{B}_{21} S^{A}_{22} S^{B}_{21} \left(1 - S^{A}_{22} S^{B}_{11}\right)^{-1} \end{bmatrix} \begin{bmatrix} a_{1A} \\ a_{2B} \end{bmatrix} \qquad (1.184)$$

Mode Matching

The main steps of the calculation are: to build multimode scattering matrix for each waveguide junction, then to apply the scattering matrix techniques to resolve characteristics of the whole structure, and then to post-process the resulted fields.

An expansion of the transverse fields $E_\perp$ and $H_\perp$ in terms of the eigenmodes e_l in the waveguide is given by:

$$\mathbf{E}_\perp = \sum_{l=1}^{M} (A_l + B_l)\mathbf{e}_l \qquad \textbf{(1.185)}$$

$$\mathbf{H}_\perp = \sum_{l=1}^{M} Y_l (A_l - B_l)\mathbf{e}_l \times \vec{\mathbf{z}} \ . \qquad \textbf{(1.186)}$$

where *A* is a modal amplitude of incident wave and *B* is the amplitude of the reflected wave (Fig.1), and *Y* is the characteristic wave admittance of the mode.

Normalisation of the eigenmodes was chosen so that the modes are orthogonal, that is:

$$\int \mathbf{e}_l \cdot \mathbf{e}_m ds = \delta_{lm} \qquad \textbf{(1.187)}$$

where δ is Kronecker delta function. Longitudinal electric field is the sum over E (TM) modes :

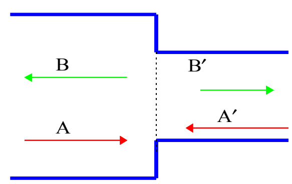
$$\mathbf{E}_z = \sum_{l=1}^{Me} i \frac{(k_l^e)^2}{\gamma_l^e} (A_l^e - B_l^e) \mathbf{e}_l z_l \qquad \textbf{(1.188)}$$

where k_l^e is cut-off wave value of the mode, γ_l^e propagation value for the particular frequency, and ez_l normalised as:

$$(k_l^e)^2 \int ez_l \cdot ez_m ds = 1. \qquad \textbf{(1.189)}$$

Derivation of the eigenmodes can be done analytically only for simple geometries such as circular or rectangular ones. So, for arbitrary cross section another method has to be applied. The method should

be able to calculate 2D flat scalar function that represents *z* - component of the field in a waveguide and correspondent cut-off frequency. After derivation of eigenmodes, applying continuity of fields in common aperture area yields the relation between incident and reflected waves i.e. scattering S matrix. Let us assume a wide to narrow transition in waveguide:



Modes in scattering matrix formulation

Firstly, we require two coupling matrices with the elements:

$$\eta_{ab}^{left} = \int e_a^{left} \cdot e_b^{right} ds \qquad \textbf{(1.190)}$$

where a = 1..M_{left}, b = 1..M_{right}; The values of M_{left} and M_{right} are governed by the relative convergence phenomena and strongly depend on the geometry.

$$\eta_{ab}^{right} = \delta_{ab} \qquad \textbf{(1.191)}$$

where *a, b* =1..M_{right}, integration over common cross-section, e_a^{left} is the a-th eigenmode of the left waveguide and e_b^{right} is b-th eigenmode of the right waveguide. The integration can be provided either analytically, for simple cross-section, or numerically, for arbitrary ones. Next step is derivation of admittance matrixes:

$$Y_{ml}^{left} = \sum_{a=1}^{Mleft} Y_a^{left} \eta_{am}^{left} \eta_{al}^{right} \qquad \textbf{(1.192)}$$

$$Y_{ml}^{right} = \sum_{b=1}^{Mright} Y_b^{right} \delta_{ml} \qquad \textbf{(1.193)}$$

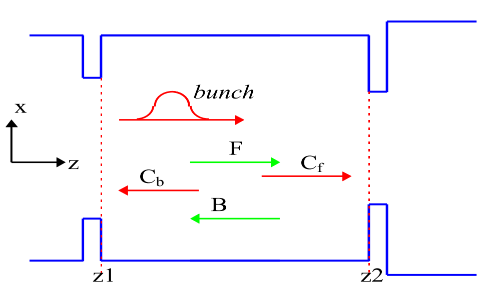
where *m, l* = 1..M$_{right}$. Equation for the full admittance matrix is $Y = Y^{left} + Y^{right}$**. Scattering matrix of the junction is:**

$$S = \begin{pmatrix} (2Y^{-1}Y^{left})^T \eta^{left} - I & (2Y^{-1}Y^{left})^T \eta^{right} \\ (2Y^{-1}Y^{right})^T \eta^{left} & (2Y^{-1}Y^{right})^T \eta^{right} - I \end{pmatrix} \quad \textbf{(1.194)}$$

Coupling Impedance

The properties of the scattering matrix approach allows one to calculate the frequency dependent impedance for 3D long structures, frequencies below and above cut-off, and to simulate non-relativistic particles. The computation time for the method does not depend on the length of the structure.

Consider a waveguide between two obstacles as shown below:



Excitation of a waveguide between two obstacles.

The beam excites waveguide modes with amplitudes:

$$C_{b,f} = \frac{1}{N_s}\int_{z1}^{z2} \vec{j}_\omega \vec{E}_{b,f} dv, \quad N_s = 2\int \vec{E}\times\vec{H} ds \qquad \textbf{(1.195)}$$

here indexes *b, f* denote the modes propagating along forward *(+z)* and backward *(-z)* direction, $\vec{j}_\omega$- beam current density, *dv* - integration over space from *z1* to *z2*, *ds* - integration over waveguide cross-section, *E* and *H* - eigenmode fields. Using (19) amplitude of modes in the waveguide for *z1* coordinate we have:

$$B = (I - S^{right}S^{left})^{-1}(S^{right}S^{left}C_b + S^{right}C_f^{z1}), \quad F = S^{left}(B + C_b) \qquad \textbf{(1.196)}$$

where S^{right} , S^{left} are scattering matrices that resulted from cascading to the right and left from *z1, C_b* - is a vector of modes induced towards *(-z)* direction, and $\left(C_f^{z1}\right)_m = \left(C_f\right)_m e^{i\gamma(z1-z2)}$ is a vector of amplitudes excited forward but converted to *z1* coordinate. Fields outside

the waveguide can be computed using (14). Longitudinal impedance for frequency ω is given by:

$$Z_l(\omega) = \int_{-\infty}^{+\infty} Ez_s(\omega) e^{-i \cdot k \cdot z} dz \qquad \textbf{(1.197)}$$

where $k = 2\pi\omega/\beta c$, βc - beam velocity.

Longitudinal electric fields is:

$Ez_s(\omega) = Ez_{mod}(\omega) + Ez_{excit}(\omega) + E_{bunch}(\omega)$, **where $Ez_{mod}(\omega)$ follows from (4), $E_{bunch} = j_\omega / i\omega$ and similar to the (4):**

$$Ez_{excit}(z) = \sum_{l=1}^{Me} i \frac{(k_l^e)^2}{\gamma_l^e} (C(z)_f - C(z)_b) ez_l \qquad \textbf{(1.198)}$$

The integration is performed along a witness bunch trajectory. The impedance for a structures with arbitrary cross-section has rather complex *(x,y)* dependence in the transverse plane. The *Ez* field is integrated along five parallel trajectories with coordinates: (x_w, y_w), (x_w+dx, y_w), (x_w, y_w+dy), (x_w-dx, y_w),

(x_w,y_w-dy). **The transverse impedance for ultra-relativistic particles is obtained by taking the transverse derivative of** ***$Z_l(\omega)$.*** **Direct integration of the transverse forces can be used for non-relativistic particles. Sum of** ***$Z_l(\omega)$*** **over all waveguides, that form the whole structure, gives coupling impedance for the structure.**

Lecture #1 Tutorial Problems

1. Measurements made on an accelerator cavity indicate that the impedance of the cavity is almost entirely inductive in nature ($Z = -i\omega L$)

(a) By taking the Fourier transform calculate the longitudinal wake function, W(s) for this case. (You may find it helpful, to use the following inverse FT $Z(\omega) = \int_{-\infty}^{\infty} W(t')\exp(i\omega t')dt'$)

(b) Take the convolution with a general time dependent current, I(t) and hence obtain the voltage.

(c) Obtain the bunch wakefield for a charge with a Gaussian distribution by taking the convolution with a Gaussian line density ($\lambda(s) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{s^2}{2\sigma_z^2}\right)$) with the wake of part a.

2. Given the fact that the wakefield $W(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)\exp(-i\omega\tau)d\omega$, is a real quantity and that for a beam traveling at the velocity of light there can be no wake ahead of the beam (causality), prove that:

(a) Re{Z(ω)} = Re{Z(-ω)} and Im{Z(ω)} = -Im{Z(-ω)}

(b) the self-wake seen by the driving particle itself is half that of the total wakefield that a trailing particle will see (hint: expand the exp(i ωt) into sin and cos functions and take the real and imaginary parts of the wake given in terms of the inverse Fourier transform of impedance, then derive an expression for the wake that does not take into account causality and compare it to one that does include it) . This is known as the *fundamental theorem of beam loading*

3. Given the longitudinal impedance (per unit length) of a lossy circular waveguide of radius, b and, conductivity, σ: $Z = \frac{1}{1+i\,\mathrm{sgn}(w)}\frac{1}{\pi b}\sqrt{\frac{|\omega| Z_0}{2c\sigma}}$, calculate the associated wake function. Here, Z_0 =377 Ohms is the impedance of free space and c is the velocity of light (hint: refer to the table 2.1 of transforms in chapter 2 of A. Chao's book).

4. (a)Derive the expression for the longitudinal loss factor of a bunch in terms of the λ(k), the Fourier transform of the line density of the bunch and the impedance of the wake Z(k) (answer: $k_{loss} = \frac{c}{\pi}\int_0^{\infty} Z(k)\,|\lambda(k)|^2 dk$).

(b) Now do the same for the transverse loss factor (often called the kick factor) for $\lambda(s) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{s^2}{2\sigma_z^2}\right)$ and the transform $\lambda(\omega) = \exp\left(-\frac{\omega^2}{2\sigma_z^2 c^2}\right)$. In this case the impedance is defined in terms of the transverse wake by: $Z_t(\omega) = -\frac{i}{c}\int_0^{\infty} W_t(s)e^{i\omega s/c}ds$ (hint: take real and imaginary parts of the wake and realize that the imaginary part of the impedance now plays the same role as the real part did in part a. Answer: $k_{t,loss} = \frac{2}{\pi^{3/2}} \int_0^{\infty} Z_r(w) D(\omega\sigma_z / c) d\omega$ where Dawson's integral is given by: $D(x) = e^{-x^2} \int_0^x e^{y^2} dy$).

In all questions, pay attention to the units of your answers. For example, you will always expect to see longitudinal impedance in units of Ohms/m and wakefields in V/C/m (or V/pC/m etc).