Course Summary Electrodynamics (PHYS30441) Terry Wyatt

1 Coordinates and Notation

As shown in Figure 1, relative to the origin O:

• Unprimed coordinates are used for the space-time coordinates of the point P at which potentials and fields are evaluated: $(\boldsymbol{r}, t)$.

• Primed coordinates are used for the space-time coordinates of the source (charge, current): $(\boldsymbol{r'}, t')$.

• Vector from the source to the position at which potentials and fields are evaluated:

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'} = (x - x')\,\widehat{\boldsymbol{x}} + (y - y')\,\widehat{\boldsymbol{y}} + (z - z')\,\widehat{\boldsymbol{z}}.$$

Magnitude: $R = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{\frac{1}{2}} = \left[r^2 + r'^2 - 2rr'\cos\alpha\right]^{\frac{1}{2}}$,
where α is the angle between $\boldsymbol{r}$ and $\boldsymbol{r'}$.

Unit vector: $\widehat{\boldsymbol{R}} = \dfrac{\boldsymbol{r} - \boldsymbol{r'}}{|\boldsymbol{r} - \boldsymbol{r'}|} = \dfrac{\boldsymbol{R}}{R}$.

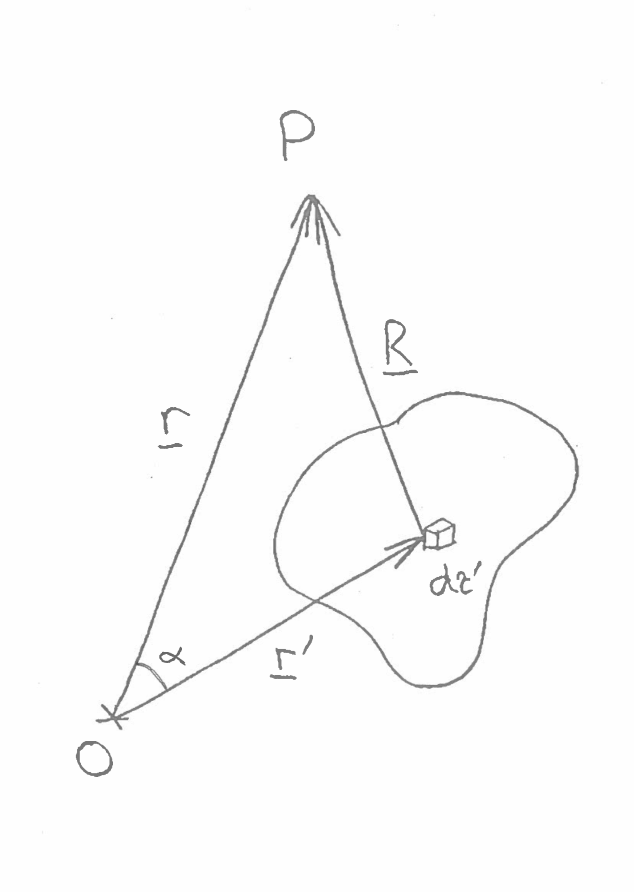

Figure 1: Diagram showing the coordinates used for the observer and sources.

The operator $\boldsymbol{\nabla}$ represents differentiation with respect to the *unprimed* coordinates:

$$\boldsymbol{\nabla} = \widehat{\boldsymbol{x}}\frac{\partial}{\partial x} + \widehat{\boldsymbol{y}}\frac{\partial}{\partial y} + \widehat{\boldsymbol{z}}\frac{\partial}{\partial z}.$$

The operator $\boldsymbol{\nabla}_{r'}$ represents differentiation with respect to the *primed* coordinates:

$$\boldsymbol{\nabla}_{r'} = \widehat{\boldsymbol{x}}\frac{\partial}{\partial x'} + \widehat{\boldsymbol{y}}\frac{\partial}{\partial y'} + \widehat{\boldsymbol{z}}\frac{\partial}{\partial z'}.$$

It is important to note that $\boldsymbol{\nabla} f(r') = \boldsymbol{\nabla}_{r'} f(r) = 0$, whereas $\boldsymbol{\nabla} f(R) = -\boldsymbol{\nabla}_{r'} f(R)$.

For example, $\boldsymbol{\nabla}\left(\frac{1}{R}\right) = -\frac{\widehat{\boldsymbol{R}}}{R^2}, \qquad \boldsymbol{\nabla}_{r'}\left(\frac{1}{R}\right) = \frac{\widehat{\boldsymbol{R}}}{R^2}.$

Note also that $\boldsymbol{\nabla}\cdot\left(\frac{\widehat{\boldsymbol{R}}}{R^2}\right) = \nabla^2\left(\frac{1}{R}\right) = -4\pi\,\delta^3(\boldsymbol{R}).$

Here $\delta^3(\boldsymbol{R})$ is the 3-dimensional delta function, which has the property that the volume integral

$$\int_{\mathcal{V}} f(\boldsymbol{r})\,\delta^3(\boldsymbol{r}-\boldsymbol{a})\,\mathrm{d}\tau' \begin{cases} = f(\boldsymbol{a}) \text{ if the volume } \mathcal{V} \text{ contains the point } \boldsymbol{r}=\boldsymbol{a}, \text{ and} \\ = 0 \text{ otherwise..} \end{cases}$$

In the case of moving sources, the retarded time at the source, t_{ret}, takes into account the time taken for a light signal to propagate a distance R:

$t_{\text{ret}} = t - \frac{R}{c}.$

Index notation for 3-vectors

It is useful to practice using cartesian index notation for 3-dimensional vectors. As well as being a useful technique in its own right, it may also help make the transition to using index notation for 4-vectors less daunting.

We write the "i" cartesian component of the 3-dimensional vector $\boldsymbol{u}$ as

$$u_i = [\boldsymbol{u}]_i\,, \qquad \text{where} \qquad i = 1, 2, 3,$$

and, in particular, we write $x_i = [\boldsymbol{r}]_i$.

Because the three cartesian coordinates are orthogonal we have

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \begin{cases} = 1 \text{ if } i = j, \\ = 0 \text{ if } i \neq j. \end{cases}$$

The *summation convention* implies that if an index is repeated in an expression that index is summed over (unless it is explicitly stated to the contrary). Thus, we can write, for example,

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{\partial u_i}{\partial x_i},$$

$$\boldsymbol{\nabla}\times\boldsymbol{u} = \widehat{\boldsymbol{x}}_{\boldsymbol{i}}\,\epsilon_{ijk}\,\frac{\partial u_k}{\partial x_j},$$

$$\boldsymbol{u}\times\boldsymbol{v} = \widehat{\boldsymbol{x}}_{\boldsymbol{i}}\,\epsilon_{ijk}\,u_j v_k,$$

where

$$\underbrace{\epsilon_{123} = \epsilon_{312} = \epsilon_{231}}_{\text{cyclic permutations}} \quad = \quad \underbrace{-\epsilon_{132} = -\epsilon_{213} = -\epsilon_{321}}_{\text{anti-cyclic permutations}}.$$

Of course, this implies $\epsilon_{ijk} = 0$ if any two indices are equal.

The following relation is useful

$$\epsilon_{ijk}\epsilon_{kmn} = \epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}.$$

2 Electrostatics and Magnetostatics

Potentials from sources:

'Point' source	$V = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{R}$	
Volume ($\mathrm{d}\tau'$)	$V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \dfrac{\rho(\boldsymbol{r'})}{R}\,\mathrm{d}\tau'$	$\boldsymbol{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\boldsymbol{j}(\boldsymbol{r'})}{R}\,\mathrm{d}\tau'$
Surface ($\mathrm{d}a'$)	$V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \dfrac{\sigma(\boldsymbol{r'})}{R}\,\mathrm{d}a'$	$\boldsymbol{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\boldsymbol{j}(\boldsymbol{r'})}{R}\,\mathrm{d}a'$
Line ($\mathrm{d}l'$)	$V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \dfrac{\lambda(\boldsymbol{r'})}{R}\,\mathrm{d}l'$	$\boldsymbol{A} = \dfrac{\mu_0 I}{4\pi}\displaystyle\int \dfrac{\mathrm{d}\boldsymbol{l'}}{R}$
Fields from potentials	$\boldsymbol{E} = -\boldsymbol{\nabla} V$	$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$
Fields from sources	$\boldsymbol{E} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\widehat{\boldsymbol{R}}}{R^2} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\boldsymbol{R}}{R^3}$	$\boldsymbol{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\boldsymbol{j}(\boldsymbol{r'})\times\widehat{\boldsymbol{R}}\,\mathrm{d}\tau'}{R^2} = \dfrac{\mu_0 I}{4\pi}\displaystyle\int \dfrac{\mathrm{d}\boldsymbol{l'}\times\widehat{\boldsymbol{R}}}{R^2}$

Inhomogeneous field equations:

Differential versions	$\boldsymbol{\nabla}\cdot\boldsymbol{E} = -\nabla^2 V = \dfrac{\rho}{\varepsilon_0}$	$\boldsymbol{\nabla}\times\boldsymbol{B} = -\nabla^2\boldsymbol{A} = \mu_0\boldsymbol{j}$
Integral versions	$\displaystyle\oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \dfrac{1}{\varepsilon_0}q_{\mathrm{enclosed}}$	$\displaystyle\oint \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l} = \mu_0 I_{\mathrm{enclosed}}$

Note: the relation $\boldsymbol{\nabla}\times\boldsymbol{B} = -\nabla^2\boldsymbol{A}$ requires choice of the Coulomb gauge: $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$

Homogeneous field equations	$\boldsymbol{\nabla}\times\boldsymbol{E} = 0$	$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$

In the static case we require $\dfrac{\partial\rho}{\partial t} = 0,\ \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0,\ \dfrac{\partial\boldsymbol{j}}{\partial t} = 0$ and $\dfrac{\partial I}{\partial t} = 0$

Notes:

1. The potentials and fields obey the *principle of superposition.* This means, for example, that many results proven for the special case of a point charge can be taken as valid for a general charge distribution (which can be built up from a collection of point charges).

2. The equations involving the vector potential, such as,

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(\boldsymbol{r'})}{R}\,\mathrm{d}\tau', \qquad -\nabla^2\boldsymbol{A} = \mu_0\boldsymbol{j}, \qquad \nabla^2\boldsymbol{A} = 0,$$

can sometimes usefully be thought as three scalar equations relating the three components of $\boldsymbol{A}$ and $\boldsymbol{j}$. Thus,

$$A_i = \frac{\mu_0}{4\pi}\int \frac{j_i(\boldsymbol{r'})}{R}\,\mathrm{d}\tau', \qquad -\nabla^2 A_i = \mu_0 j_i, \qquad \nabla^2 A_i = 0,$$

where $i = x, y, z$ or $i = 1, 2, 3$.

3. Using Stokes's theorem and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ it can be shown that:

$$\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}.$$

2.1 Solutions to Laplace's equation

In regions with no sources, Poisson's equation $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ reduces to Laplace's equation $\nabla^2 V = 0$.

Solutions to Laplace's equation have the following properties:

- The value of $V(\boldsymbol{r})$ is equal to the average value of V over a spherical surface centred at $\boldsymbol{r}$.
  $\Longrightarrow$ No local maxima or minima can be present[1] *within* a region satisfying $\boldsymbol{\nabla}^2 V = 0$.
  $\Longrightarrow$ Earnshaw's Theorem: A charged particle cannot be held in a position of stable equilibrium by electrostatic forces alone.
- The particular solution to Laplace's equation requires knowledge of the *boundary conditions.*
- **Uniqueness Theorem:** There is only one unique solution to Laplace's equation that satisfies a given complete set of boundary conditions.

The solutions to Laplace's equation in many problems employ the *separation of variables*:

Cartesian coordinates:

$V(x, y, z) = X(x)Y(y)Z(z)$ where the most convenient choice for the general solution can be either

- $X(x) = A\sin(kx) + B\cos(kx)$ for coordinates in which the problem is bounded, or
- $X(x) = Ae^{kx} + Be^{-kx}$ for coordinates in which the problem is unbounded.

Spherical polar coordinates with azimuthal (ϕ) symmetry:

$V(r, \theta) = \displaystyle\sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$ where the first three Legendre polynomials are given by

$$\begin{aligned} P_0(\cos\theta) &= 1, \\ P_1(\cos\theta) &= \cos\theta, \\ P_2(\cos\theta) &= \left(3\cos^2\theta - 1\right)/2, \\ P_l(\cos\theta) &= \frac{1}{2^l l!} \left(\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)} \right)^l \left(\cos^2\theta - 1\right)^l. \end{aligned}$$

Notes on boundary conditions:

1. In most problems the complete set of boundary conditions is not stated explicitly and has to be inferred on physical grounds. It is important that you can use confidently the techniques for determining boundary conditions that you will have met in 2nd year E&M.

[1] Turning points in V as a function of one or more coordinates are allowed within the volume. However, maxima (or minima) in all three coordinates at the same point in space within the volume are not allowed.

2. For example, in electrostatics:

   (a) When considering a charge distribution of finite spatial extent its contribution to $V \to 0$ as $R \to \infty$ (far from the charge distribution).

   (b) Other boundary conditions relevant to a surface charge density σ are that:

      i. the scalar potential V must be continuous as one crosses the boundary,

      ii. the component of the $\boldsymbol{E}$ field parallel to the surface must be continuous,

      iii. the component of the $\boldsymbol{E}$ field perpendicular to the surface is discontinuous by an amount $\Delta E_\perp = E_\perp^{\text{out}} - E_\perp^{\text{in}} = \sigma/\epsilon_0$. (See Figure 2 [left]).

   Similarly, in magnetostatics:

   (a) When considering a current distribution of finite spatial extent its contribution to $\boldsymbol{A} \to 0$ as $R \to \infty$ (far from the current distribution).

   (b) When considering a surface current density j:

      i. the vector potential $\boldsymbol{A}$ must be continuous as one crosses the boundary,

      ii. the component of the $\boldsymbol{B}$ field perpendicular to the surface must be continuous,

      iii. the component of the $\boldsymbol{B}$ field parallel to the surface is discontinuous by an amount $\Delta B_\parallel = B_\parallel^{\text{above}} - B_\parallel^{\text{below}} = \mu_0 j$. (See Figure 2 [right]).

3. Sometimes a solution is considered that has to be valid only within a particular region of space (for example, inside or outside of a given closed surface). It is very important to remember that the physical constraint that V or $\boldsymbol{A}$ remain finite as $R \to 0$ can be applied only if the region of space being considered actually contains the point $R = 0$! A similar consideration applies when considering the physical constraint that V or $\boldsymbol{A}$ remain finite as $R \to \infty$.

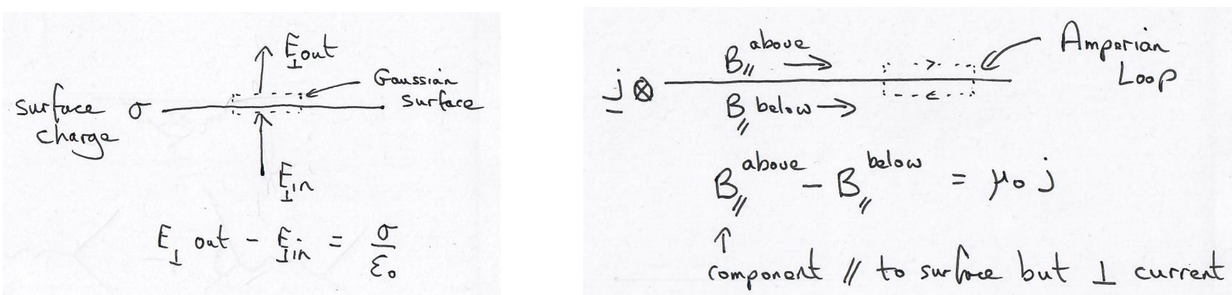

Figure 2: Diagrams showing [left] gaussian pill box used to calculate the discontinuity in $E_\perp$ due to the presence of a surface charge density σ, and [right] amperian loop used to calculate the discontinuity in $B_\parallel$ due to the presence of a surface current density j.

2.2 Multipole expansions

It is sometimes useful to evaluate approximate expressions for the potentials and/or fields at distances from the sources that are much greater than the spatial extent of the sources ($r \gg r'$). For example, sometimes we can have quite complicated-looking distributions of charges (or currents), but the most important physical features of the resulting potentials and/or fields can be described to a reasonable approximation by, say, a dipole. Particularly in magnetostatics, where the "monopole" term is always zero, working out the dipole moment is a particularly convenient way of finding the potentials and/or fields resulting from most current distributions.

The following results for V and $\boldsymbol{A}$ derive from the expansion of $\frac{1}{R}$ in powers of $\frac{r'}{r}$ and Legendre Polynomials

$$\frac{1}{R} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(\boldsymbol{r}')\,\mathrm{d}\tau' \\ &= \frac{1}{4\pi\varepsilon_0}\Bigg[\underbrace{\frac{1}{r}\int\rho(\boldsymbol{r}')\,\mathrm{d}\tau'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\int r'\cos\alpha\,\rho(\boldsymbol{r}')\,\mathrm{d}\tau'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\boldsymbol{r}')\,\mathrm{d}\tau'}_{\text{quadrupole}} + \dots\Bigg] \end{aligned}$$

The electric "monopole" term is essentially just Coulomb's law, with the approximation that the entire charge density is collapsed to the origin.

$$V(\boldsymbol{r})_{\text{monopole}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r},$$

where Q is the total charge.

The electric "dipole" term may be written as

$$V(\boldsymbol{r})_{\text{dipole}} = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\widehat{\boldsymbol{r}}}{r^2},$$

where the *electric dipole moment*, $\boldsymbol{p}$, is given by:

$$\boldsymbol{p} = \int \boldsymbol{r}'\rho(\boldsymbol{r}')\,\mathrm{d}\tau'.$$

$\boldsymbol{p}$ depends only on the charge distribution and not on $\boldsymbol{r}$. If we choose coordinates such that the dipole is at the origin and points along the z direction, $\boldsymbol{p} = p\widehat{\boldsymbol{z}}$, the electric field is given by

$$\boldsymbol{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\widehat{\boldsymbol{r}} + \sin\theta\,\widehat{\boldsymbol{\theta}}\right).$$

We can make a similar multipole expansion for the vector potential far away from a localised current density distribution.

$$\begin{aligned} A(\boldsymbol{r}) &= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\,\mathrm{d}\boldsymbol{l}' \\ &= \frac{\mu_0 I}{4\pi}\Bigg[\underbrace{\frac{1}{r}\oint\mathrm{d}\boldsymbol{l}'}_{\text{zero}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\,\mathrm{d}\boldsymbol{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\mathrm{d}\boldsymbol{l}'}_{\text{quadrupole}} + \dots\Bigg] \end{aligned}$$

The vector potential for a magnetic dipole is given by

$$A(\boldsymbol{r})_{\text{dipole}} = \frac{\mu_0 I}{4\pi}\frac{\boldsymbol{a}\times\widehat{\boldsymbol{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\widehat{\boldsymbol{r}}}{r^2},$$

where $\boldsymbol{m} = I\boldsymbol{a}$ is the *magnetic dipole moment* and $\boldsymbol{a}$ is the vector area of the current loop. If the loop lies in a plane then $|\boldsymbol{a}|$ is just the area enclosed by the loop. The direction of $\boldsymbol{a}$ is given from the circulation of $\boldsymbol{I}$, e.g., by the right-hand rule.

For a magnetic dipole centred at the origin and pointing along the z direction, $\boldsymbol{m} = m\widehat{\boldsymbol{z}}$, the magnetic field is given by

$$\boldsymbol{B}(\boldsymbol{r})_{\text{dipole}} = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\widehat{\boldsymbol{r}} + \sin\theta\,\widehat{\boldsymbol{\theta}}\right]$$

3 Electrodynamics

Inhomogeneous field equations $\quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\varepsilon_0 \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \varepsilon_0 \mu_0 \dfrac{\partial \boldsymbol{E}}{\partial t}$

Homogeneous field equations $\quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$

Integral relations $\quad \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\dfrac{\partial \Phi_m}{\partial t} \quad \oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I_{\text{enclosed}} + \varepsilon_0 \mu_0 \dfrac{\partial \Phi_E}{\partial t}$

where $\quad \Phi_m = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \quad \Phi_E = \int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a}$

and the surface S is enclosed by the line over which the closed line integral is performed

Lorentz force $\quad \boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$

Fields from potentials $\quad \boldsymbol{E} = -\boldsymbol{\nabla} V - \dfrac{\partial \boldsymbol{A}}{\partial t} \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

The gauge transformation ... $\quad V \Longrightarrow V - \dfrac{\partial \psi}{\partial t} \quad \boldsymbol{A} \Longrightarrow \boldsymbol{A} + \boldsymbol{\nabla}\psi$

... leaves the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ unchanged

Choosing the Lorenz gauge condition ... $\quad \dfrac{1}{c^2}\dfrac{\partial V}{\partial t} + \boldsymbol{\nabla} \cdot A = 0, \quad \text{where } \dfrac{1}{c^2} = \varepsilon_0 \mu_0$

... yields the inhomogeneous wave equations for the potentials, in the form ... $\quad \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2\right) V = \dfrac{\rho}{\varepsilon_0} \quad \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2\right) \boldsymbol{A} = \mu_0 \boldsymbol{j}$

or $\quad \Box^2 V = \dfrac{\rho}{\varepsilon_0} \quad \Box^2 \boldsymbol{A} = \mu_0 \boldsymbol{j}$

where $\quad \Box^2 = \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2\right)$

Integral solutions to the inhomogeneous wave equations for the potentials

When the source (charge distribution) is moving we must take into account the time $(\frac{R}{c})$ taken for a signal travelling at the speed of light c to propagate a distance R from the source to the point $P(\boldsymbol{r}, t)$ at which we evaluate the potential.

This allows us to generalise the potentials we wrote down previously for the static case to:

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{r'}, t_{\text{ret}})}{R} \mathrm{d}\tau' \qquad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{r'}, \boldsymbol{t}_{\textbf{ret}})}{R} \mathrm{d}\tau' = \frac{\mu_0}{4\pi} \int \frac{\rho(\boldsymbol{r'}, t_{\text{ret}})\, \boldsymbol{v}(\boldsymbol{r'}, \boldsymbol{t}_{\textbf{ret}})}{R} \mathrm{d}\tau'$$

where we need to evaluate the charge density ρ, the current density $\boldsymbol{j}$, charge velocity $\boldsymbol{v}$, and the distance R, all at the *retarded* time, $t_{\text{ret}} = t - \dfrac{R}{c}$, and position, $\boldsymbol{r'}$, at the source.

These potentials satisfy the Wave Equation and the Lorenz gauge condition.

Local Conservaton Laws and Symmetries in Electrodynamics

In special relativity all observers agree that two space-time events occur at the same point in time, i.e., that they are *simultaneous*, **only** if they occur also at the same point in *space*. Therefore we require the laws of physics to respect *local* conservation rules, such as the continuity equation (local conservation of charge) and the local conservation of energy.

Conservation of electric charge $\quad \dfrac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$

Conservation of energy $\quad \boldsymbol{E} \cdot \boldsymbol{v}\rho = \boldsymbol{E} \cdot \boldsymbol{j} = -\boldsymbol{\nabla} \cdot \boldsymbol{S} - \dfrac{\partial u}{\partial t}$

where:

$\boldsymbol{E} \cdot \boldsymbol{j}$ is the rate of change of energy density for charge ρ due to work done by $\boldsymbol{E}$ field

$\boldsymbol{S} = \dfrac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$ is the energy flux density in the fields (Poynting vector)

$u = \dfrac{\varepsilon_0}{2} \boldsymbol{E}^2 + \dfrac{1}{2\mu_0} \boldsymbol{B}^2$ is the energy density in the fields

In a similar vein, the local "gauge symmetry" that leaves the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ unchanged under the gauge transformation $V \Longrightarrow V - \dfrac{\partial \psi}{\partial t}$, $\boldsymbol{A} \Longrightarrow \boldsymbol{A} + \boldsymbol{\nabla}\psi$, implies that ψ can be chosen independently at each point in space-time, subject to satisfying the chosen gauge condition (such as the Lorenz gauge condition).

4 Electrodynamics in Lorentz-Covariant Notation

Special Relativity

In an *intertial frame of reference* a body with no nett force acting upon it does not accelerate.

Einstein's Postulates

I The laws of physics (results of experiments) are the same in all inertial frames of reference.

II The speed of light (in a vacuum) is the same in all inertial frames — this represents the maximum possible speed for any physical entity.

The Space-Time 4-Vector

Contravariant (upper index) $x^\mu = [x^0, x^1, x^2, x^3] = [ct, x, y, z] = [ct, \boldsymbol{r}]$

Covariant (lower index) $x_\mu = [x_0, x_1, x_2, x_3] = [x^0, -x^1, -x^2, -x^3] = [ct, -x, -y, -z] = [ct, -\boldsymbol{r}]$

Frame S' (primed coordinates) moves relative to frame S (unprimed coordinates) in the x^1 direction with constant speed β (in units of c).

At $x'^0 = x^0 = 0$ we have $x'^1 = x^1 = 0$.

Lorentz Transformations

$$\begin{aligned} x'^0 &= \gamma\,(x^0 - \beta x^1) \\ x'^1 &= \gamma\,(-\beta x^0 + x^1) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned} \qquad \begin{aligned} x'_0 &= \gamma\,(x_0 + \beta x_1) \\ x'_1 &= \gamma\,(\beta x_0 + x_1) \\ x'_2 &= x_2 \\ x'_3 &= x_3 \end{aligned}$$

$$\begin{aligned} x_0 &= \gamma\,(x'_0 - \beta x'_1) \\ x_1 &= \gamma\,(-\beta x'_0 + x'_1) \\ x_2 &= x'_2 \\ x_3 &= x'_3 \end{aligned} \qquad \begin{aligned} x^0 &= \gamma\,(x'^0 + \beta x'^1) \\ x^1 &= \gamma\,(\beta x'^0 + x'^1) \\ x^2 &= x'^2 \\ x^3 &= x'^3 \end{aligned}$$

and where
$$\gamma = \frac{1}{\left(1-(v/c)^2\right)^{\frac{1}{2}}} = \frac{1}{(1-\beta^2)^{\frac{1}{2}}} = \left(1-\beta^2\right)^{-\frac{1}{2}}$$

Matrix form
$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \qquad x^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu\, x'^\nu$$

$$x_\mu = \Lambda^\nu{}_\mu\, x'_\nu \qquad x'_\mu = \left(\Lambda^{-1}\right)^\nu{}_\mu\, x_\nu$$

where
$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left(\Lambda^{-1}\right)^\nu{}_\mu = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and also
$$\Lambda^\mu{}_\nu = \frac{\mathrm{d}x'^\mu}{\mathrm{d}x^\nu} = \frac{\mathrm{d}x_\nu}{\mathrm{d}x'_\mu} \qquad \left(\Lambda^{-1}\right)^\nu{}_\mu = \frac{\mathrm{d}x^\nu}{\mathrm{d}x'^\mu} = \frac{\mathrm{d}x'_\mu}{\mathrm{d}x_\nu}$$

where
$$\Lambda^\mu{}_\gamma \left(\Lambda^{-1}\right)^\gamma{}_\nu = \left(\Lambda^{-1}\right)^\mu{}_\gamma \Lambda^\gamma{}_\nu = \delta^\mu_\nu \qquad \delta^\mu_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Metric tensor
$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \begin{aligned} x^\mu &= g^{\mu\nu} x_\nu \\ x_\mu &= g_{\mu\nu} x^\nu \end{aligned}$$

and
$$g^{\mu\gamma} g_{\gamma\nu} = g_{\mu\gamma} g^{\gamma\nu} = \delta^\mu_\nu$$

Scalar product (Lorentz invariant)
$$\begin{aligned} a_\mu b^\mu = a^\mu b_\mu = a_\mu\, g^{\mu\nu} b_\nu = a^\mu g_{\mu\nu}\, b^\nu &= a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3 \\ &= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \end{aligned}$$

Magnitude (Lorentz invariant)
$$x_\mu x^\mu = x^\mu x_\mu = x_\mu\, g^{\mu\nu} x_\nu = x^\mu g_{\mu\nu}\, x^\nu = (ct)^2 - x^2 - y^2 - z^2 = (ct)^2 - r^2$$

Space-Time Intervals and Proper Time

Note: if we can show that the four components of any object obey the Lorentz Transformations when moving from one inertial frame to another this proves the object is a 4-vector. The converse is also true: if we know that an object is a 4-vector this guarantees that its components obey the Lorentz Transformations when moving from one inertial frame to another, and guarantees all of the other general properties of 4-vectors (such as that the object has a Lorentz-invariant magnitude).

The space-time interval between two events (1) and (2) is a 4-vector: $\Delta x^\mu = x^\mu_{(1)} - x^\mu_{(2)} = (c\Delta t, \Delta \boldsymbol{r})$ with magnitude given by $\Delta x^\mu \Delta x_\mu = (c\Delta t)^2 - (\Delta r)^2$

Space-time intervals are classified according to the sign of $\Delta x^\mu \Delta x_\mu$:

Time-like	$\Delta x^\mu \Delta x_\mu > 0$	The two events can be connected by a signal travelling at $v < c$.
Light-like	$\Delta x^\mu \Delta x_\mu = 0$	The two events can be connected by a signal travelling at $v = c$.
Space-like	$\Delta x^\mu \Delta x_\mu < 0$	The two events cannot be connected by a signal travelling at $v \leq c$.

For time-like intervals we can write

$$\Delta x^\mu \Delta x_\mu = (c\Delta\tau)^2,$$

where $\Delta\tau$ is the (Lorentz-invariant) *proper time interval*, as measured by a clock in the frame in which the two events happen at the same point in space. In all other frames the time interval between the two events is given by $\Delta t = \gamma\Delta\tau$. (That is, considering all possible inertial frames of reference, the proper time gives the smallest possible time interval between two space-time events).

Lorentz Transformations of Velocities and Accelerations

Velocity is a 3-dimensional vector, but does not directly constitute the 3-vector part of a 4-vector. Therefore velocity does not transform directly according to the Lorentz transformation. However, we can use the Lorentz transformation to work out how the components of velocity do transform from one frame to another.

Let us consider a particle that is travelling with velocity (in units of c) $\boldsymbol{\beta'}$ in an inertial frame of reference S'. Let frame S' be moving with velocity[2] $\boldsymbol{\beta}_{\text{LT}}$ relative to a second frame S.

What is the velocity, $\boldsymbol{\beta}$, of the particle, as measured in frame S? We shall consider two special cases:

$\boldsymbol{\beta'} \perp \boldsymbol{\beta}_{\text{LT}}$: for which we find the result $\boldsymbol{\beta} = \left(\beta_{\text{LT}}, \dfrac{\beta'}{\gamma_{\text{LT}}}\right)$, where the first component is along the direction of relative motion of the two frames and the second component is in the direction of $\boldsymbol{\beta'}$, and

$\boldsymbol{\beta'} \parallel \boldsymbol{\beta}_{\text{LT}}$: for which we find the result $\beta = \dfrac{\beta' + \beta_{\text{LT}}}{1 + \beta'\beta_{\text{LT}}}$.

These not so simple transformations motivate us to define the 4-velocity, which does transform as a 4-vector (see next section).

Similarly, acceleration does not directly constitute the 3-vector part of a 4-vector. Let now us consider a particle that has an acceleration (in units of c) $\dot{\boldsymbol{\beta}}'$ in our inertial frame S'. What is the magnitude of the acceleration, $\dot\beta$, of the particle, as measured in frame S? Again, we shall consider two special cases:

[2] We use the symbol $\boldsymbol{\beta}_{\text{LT}}$ here just to avoid any possible confusion between the speed of relative motion between the two frames and the speed of a particle within any particular frame. Similarly we shall write γ_{LT} for the γ-factor that arises from the relative motion between the two frames of reference, which would be expressed in terms of β_{LT}.

$\dot{\boldsymbol{\beta}}' \perp \boldsymbol{\beta}_{\text{LT}}$: for which we find the result $\dot{\beta} = \dfrac{\dot{\beta}'}{\gamma_{\text{LT}}^2}$, and

$\dot{\boldsymbol{\beta}}' \parallel \boldsymbol{\beta}_{\text{LT}}$: for which we find the result $\dot{\beta} = \dfrac{\dot{\beta}'}{\gamma_{\text{LT}}^3 \left(1 + \beta' \beta_{\text{LT}}\right)^3}$.

When we discuss radiation from accelerating charged particles, we shall consider the case that the particle is approximately at rest in frame S', in which case $\beta' \ll 1$ and the latter result simplifies to:

$\dot{\boldsymbol{\beta}}' \parallel \boldsymbol{\beta}_{\text{LT}}$: $\quad \dot{\beta} = \dfrac{\dot{\beta}'}{\gamma_{\text{LT}}^3}$ for the case $\beta' \ll 1$.

Other 4-Vectors

name	definition	magnitude
4-differentials	$\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left[\dfrac{1}{c}\dfrac{\partial}{\partial t}, \boldsymbol{\nabla}\right]$ $\partial^\mu = \dfrac{\partial}{\partial x_\mu} = \left[\dfrac{1}{c}\dfrac{\partial}{\partial t}, -\boldsymbol{\nabla}\right]$	$\partial^\mu \partial_\mu = \Box^2 = \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2\right)$
4-velocity	$u^\mu = \dfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = [\gamma c, \gamma \boldsymbol{v}] = \gamma c\,[1, \boldsymbol{\beta}]$	$u^\mu u_\mu = c^2$

which transforms as a 4-vector because $\mathrm{d}x^\mu$ is a 4-vector and $\mathrm{d}\tau$ is a Lorentz invariant.

name	definition	magnitude
4-momentum	$p^\mu = m u^\mu = m\dfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = [\gamma m c, \gamma m \boldsymbol{v}] = \left[\dfrac{\mathcal{E}}{c}, \boldsymbol{p}\right]$	$p^\mu p_\mu = \left(\dfrac{\mathcal{E}}{c}\right)^2 - p^2 = (mc)^2$

where m is the *rest mass* or *invariant mass* of a particle and $\mathcal{E}$ is its total energy.

name	definition	magnitude
4-current density	$j^\mu = \rho_0 u^\mu = [\gamma \rho_0 c, \gamma \rho_0 \boldsymbol{v}] = [\rho c, \rho \boldsymbol{v}]$	$j^\mu j_\mu = (\rho_0 c)^2$

where ρ_0 is the (invariant) charge density in the rest frame of the charge and $\rho = \gamma \rho_0$ is the charge density in a frame in which the charge is moving.

name	definition	magnitude
4-potential	$A^\mu = \left[\dfrac{V}{c}, \boldsymbol{A}\right]$	$A^\mu A_\mu = \left(\dfrac{V}{c}\right)^2 - \boldsymbol{A}^2$

It should be noted that:

- differentiation with respect to the *covariant* space-time coordinate x_μ transforms as a *contravariant* ∂^μ
$$\frac{\partial}{\partial x'_\mu} = \partial'^\mu = \Lambda^\mu{}_\nu \, \partial^\nu \qquad \text{and} \qquad \frac{\partial}{\partial x_\mu} = \partial^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu \, \partial'^\nu.$$
- differentiation with respect to the *contravariant* space-time coordinate x^μ transforms as a *covariant* ∂_μ
$$\frac{\partial}{\partial x^\mu} = \partial_\mu = \Lambda^\nu{}_\mu \, \partial'_\nu \qquad \text{and} \qquad \frac{\partial}{\partial x'^\mu} = \partial'_\mu = \left(\Lambda^{-1}\right)^\nu{}_\mu \, \partial_\nu.$$

A note on ultra-relativistic particles

As $v \to c,\ \beta \to 1$, the contribution of the rest mass of a particle to its energy can be neglected and we can write $\mathcal{E} \approx pc$.

In particular, for a photon we can write

$$p^\mu = [p, \boldsymbol{p}] \qquad \text{and} \qquad p^\mu p_\mu = 0.$$

The Laws of Electrodynamics Translated into Lorentz-Covariant Notation

Continuity equation	$\partial_\mu j^\mu = 0$	$\dfrac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$
Gauge transformation	$A^\mu \Longrightarrow A^\mu - \partial^\mu \psi$	$\begin{cases} V \Longrightarrow V - \dfrac{\partial \psi}{\partial t} \\ \boldsymbol{A} \Longrightarrow \boldsymbol{A} + \boldsymbol{\nabla}\psi \end{cases}$
Lorenz gauge condition	$\partial_\mu A^\mu = 0$	$\dfrac{1}{c^2}\dfrac{\partial V}{\partial t} + \boldsymbol{\nabla} \cdot A = 0$
Inhomogeneous wave equations (in the Lorenz gauge)	$\partial^\nu \partial_\nu A^\mu = \Box^2 A^\mu = \mu_0 j^\mu$	$\begin{cases} \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2\right) V = \dfrac{\rho}{\varepsilon_0} \\ \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2\right) \boldsymbol{A} = \mu_0 \boldsymbol{j} \end{cases}$
Integral solutions to the inhomogeneous wave equations for the potentials	$A^\mu = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{j^\mu(\boldsymbol{r'}, t_{\text{ret}})}{R}\, \mathrm{d}\tau'$	$\begin{cases} V(\boldsymbol{r}, t) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int \dfrac{\rho(\boldsymbol{r'}, t_{\text{ret}})}{R}\, \mathrm{d}\tau' \\ \boldsymbol{A}(\boldsymbol{r}, t) = \dfrac{\mu_0}{4\pi} \displaystyle\int \dfrac{\boldsymbol{j}(\boldsymbol{r'}, \boldsymbol{t_{\text{ret}}})}{R}\, \mathrm{d}\tau' \end{cases}$
Fields from potentials	$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$	$\begin{cases} \boldsymbol{E} = -\boldsymbol{\nabla} V - \dfrac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \end{cases}$
Inhomogeneous field equations	$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$	$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\varepsilon_0 \\ \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \varepsilon_0 \mu_0 \dfrac{\partial \boldsymbol{E}}{\partial t} \end{cases}$
Homogeneous field equations	$\partial^\mu F^{\nu\lambda} + \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} = 0$	$\begin{cases} \boldsymbol{\nabla} \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \end{cases}$
Interaction of a charged particle q with the fields	$\dfrac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = q\, F^{\mu\nu}\, u_\nu$	$\begin{cases} \dfrac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\boldsymbol{u} \cdot \boldsymbol{E} \\ \dfrac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \end{cases}$

The electromagnetic field tensor may be written in matrix form as

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{bmatrix}$$

$F^{\mu\nu}$ transforms as a 2nd rank tensor:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta} \qquad\qquad F^{\mu\nu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\alpha}\left(\Lambda^{-1}\right)^{\nu}{}_{\beta}F'^{\alpha\beta}$$

or, alternatively,

$$F' = \Lambda F \Lambda^T \qquad\qquad F = \left(\Lambda^{-1}\right) F' \left(\Lambda^{-1}\right)^T,$$

from which it can be shown that:

$$\frac{E_1'}{c} = \frac{E_1}{c} \qquad B_1' = B_1 \qquad\qquad \frac{E_1}{c} = \frac{E_1'}{c} \qquad B_1 = B_1'$$

$$\frac{E_2'}{c} = \gamma\left(\frac{E_2}{c} - \beta B_3\right) \qquad B_2' = \gamma\left(B_2 + \beta\frac{E_3}{c}\right) \qquad\qquad \frac{E_2}{c} = \gamma\left(\frac{E_2'}{c} + \beta B_3'\right) \qquad B_2 = \gamma\left(B_2' - \beta\frac{E_3'}{c}\right)$$

$$\frac{E_3'}{c} = \gamma\left(\frac{E_3}{c} + \beta B_2\right) \qquad B_3' = \gamma\left(B_3 - \beta\frac{E_2}{c}\right) \qquad\qquad \frac{E_3}{c} = \gamma\left(\frac{E_3'}{c} - \beta B_2'\right) \qquad B_3 = \gamma\left(B_3' + \beta\frac{E_2'}{c}\right)$$

Notes:

1. Starting from the transformation equations (from S to S') the inverse transformation for the fields (from S' to S) may be obtained by setting $\beta \longrightarrow -\beta$.
2. The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are 3-D vectors that are not components of a 4-vector. For the purposes of this course, I make no distinction between "upper" and "lower" indices for such 3-D vectors.

The elements of $F_{\mu\nu} = g_{\mu\alpha}F^{\alpha\beta}g_{\beta\nu}$ are the same as those of $F^{\mu\nu}$ except for the replacement $E_i \to -E_i$. (The signs of the B_i terms are unchanged.) That is,

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{bmatrix}.$$

An Example of Applying the Lorentz Transformations: Potentials and Fields for a Point Charge Moving with Constant Velocity

We consider a point charge at rest at the origin in the inertial frame S', which is moving with a *constant* velocity along the x^1 axis with speed β (in units of c) relative to the inertial frame S. (See Figure 3.)

The 4-potential A'^{μ} in S' is given by

$$A'^{0} = \frac{V'}{c} = \frac{q}{4\pi\varepsilon_0 c}\frac{1}{R'} \qquad\qquad A'^{1} = A'^{2} = A'^{3} = 0,$$

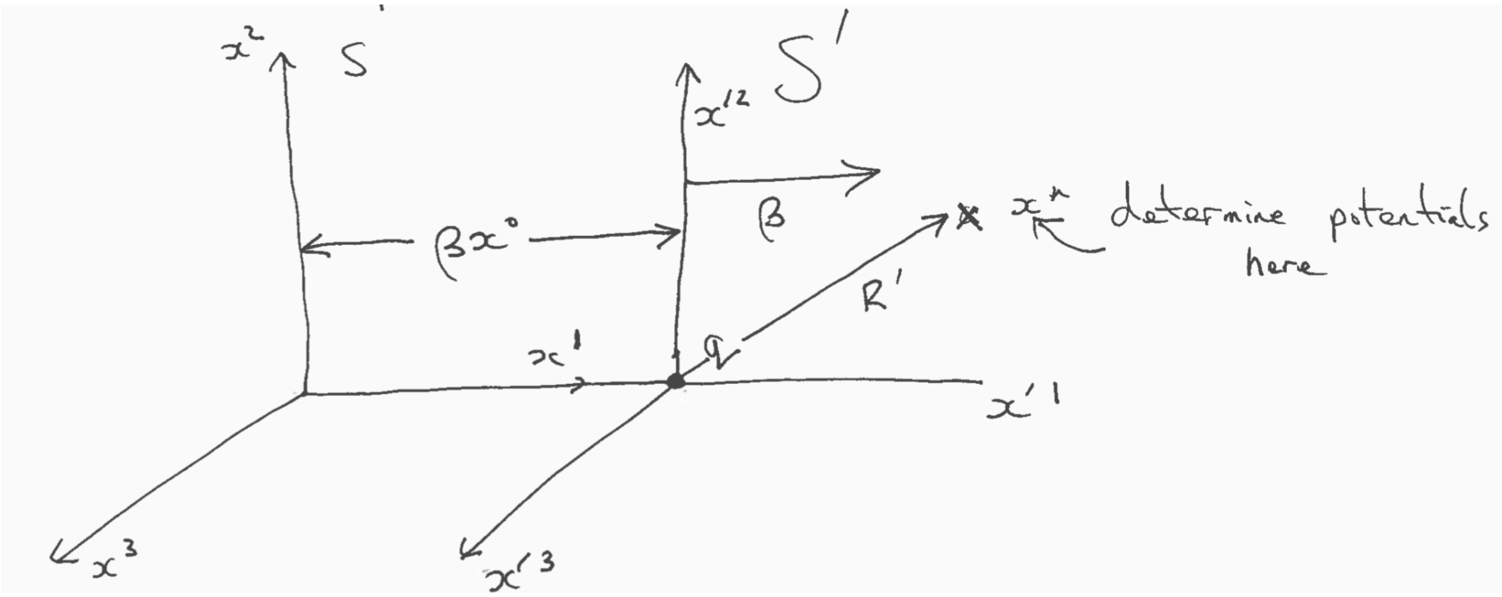

Figure 3: Sketch of the frames of reference for a point charge moving with a constant velocity.

where

$$(R')^2 = \left(x'^1\right)^2 + \left(x'^2\right)^2 + \left(x'^3\right)^2 = \left(\gamma\left[x^1 - \beta x^0\right]\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$$

N.B. In terms of (ct, x, y, z) *we could write,* $(R')^2 = (x')^2 + (y')^2 + (z')^2 = (\gamma\left[x - \beta ct\right])^2 + y^2 + z^2$, *but in this treatment I'll be sticking to* (x^0, x^1, x^2, x^3).

The (inverse) Lorentz transformation applied to the 4-potential A'^μ in S' gives the following 4-potential A^μ in frame S:

$$\begin{aligned} A^0 &= \frac{V}{c} = \gamma A'^0 = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{R'} \\ &= \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{\left[\left(\gamma\left[x^1 - \beta x^0\right]\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right]^{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned} A^1 &= \gamma\beta A'^0 = \beta A^0 = \beta\frac{V}{c} \\ &= \frac{q}{4\pi\varepsilon_0 c}\gamma\beta\frac{1}{\left[\left(\gamma\left[x^1 - \beta x^0\right]\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right]^{\frac{1}{2}}} \end{aligned}$$

$$A^2 = A^3 = 0$$

Physically, we can think of these as the potentials measured as a function of time x^0 by an observer at rest in frame S at the point x^1, x^2, x^3 (i.e., x^1, x^2, x^3 are independent of x^0).

The potentials may be expressed also as

$$A^0 = \frac{q}{4\pi\varepsilon_0 cR}\left(\frac{1}{1 - \beta^2\sin^2\theta}\right)^{\frac{1}{2}}, \qquad A^1 = \frac{\beta q}{4\pi\varepsilon_0 cR}\left(\frac{1}{1 - \beta^2\sin^2\theta}\right)^{\frac{1}{2}},$$

where $\boldsymbol{R}$ is the vector from the current position $x^1 = \beta x^0, x^2 = 0, x^3 = 0$ of the point charge to the point at which the potential is evaluated. The magnitude of R is given by

$$R^2 = \left[x^1 - \beta x^0\right]^2 + \left(x^2\right)^2 + \left(x^3\right)^2.$$

θ is the angle between $\boldsymbol{R}$ and the direction of motion.

Sketches of equipotentials in A^0 are given in Figure 4.

Developing our intuition:

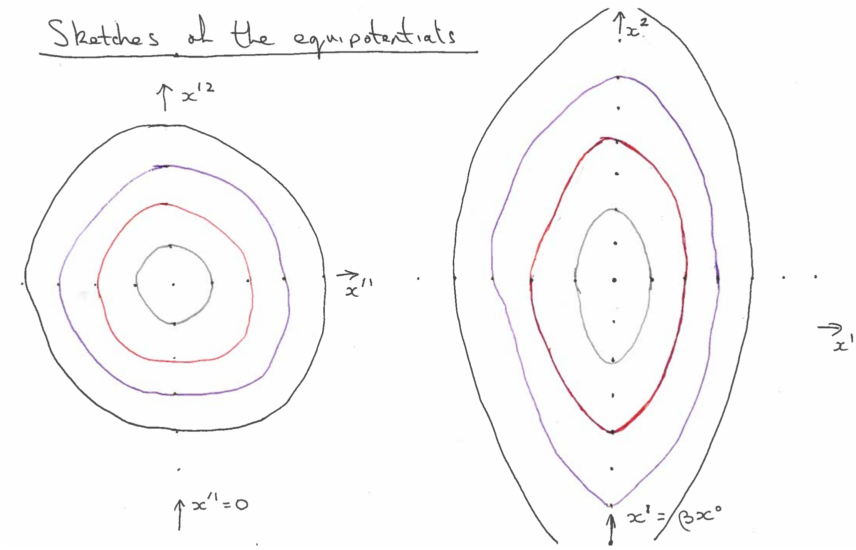

Figure 4: Sketches of equipotentials in A^0 for a point charge: [left] in the rest frame, and [right] in a frame in which the charge is moving with $\gamma \approx 2$.

1) It is perhaps surprising that the equipotentials at time x^0 are centred at the "current position" of the charge, $x^1 = \beta x^0$, rather than at the "retarded position" appropriate for a particular observation point?

2) At a fixed distance R' (as measured in the rest frame) from the charge, the scalar potential is multiplied by a factor of γ when we transform to a frame in which the charge is moving. However, most of the time we are interested in seeing how the potential varies as a function of R (distance from the charge as measured in the frame in which it is moving):

- Let's consider first displacements along the direction of motion: because distances along the direction of motion are Lorentz contracted we get a factor of γ in the denominator, which cancels the factor of γ in the numerator. Therefore, in this direction the contours in the potential fall off with R in Figure 4 in exactly the same way as they would for a stationary charge at the current position.
- Distances transverse to the direction of motion are not Lorentz contracted and therefore the contours in the potential fall off with R in Figure 4 a factor of γ more slowly than they would for a stationary charge at the current position.

In the rest frame S' of the charge, $\boldsymbol{B'} = 0$ and the field $\boldsymbol{E'}$ is given by

$$\frac{\boldsymbol{E'}}{c} = \frac{q}{4\pi\varepsilon_0 c} \frac{\boldsymbol{R'}}{(R')^3}$$

or, alternatively

$$\frac{E'_1}{c} = \frac{q}{4\pi\varepsilon_0 c} \frac{x'^1}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c} \frac{\gamma\,[x^1 - \beta x^0]}{(R')^3}$$

$$\frac{E'_2}{c} = \frac{q}{4\pi\varepsilon_0 c} \frac{x'^2}{(R')^3}$$

$$\frac{E'_3}{c} = \frac{q}{4\pi\varepsilon_0 c} \frac{x'^3}{(R')^3}.$$

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields for a point charge moving with a constant velocity along the x axis can be found in two ways: (i) from the 4-potential in the frame S in which the charge is moving, and (ii) by applying the (inverse) transformation to the fields in the rest frame S'.

Both methods give the $\boldsymbol{E}$ field as

$$\frac{E_1}{c} = \frac{q}{4\pi\varepsilon_0 c}\frac{\gamma\left[x^1-\beta x^0\right]}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{\left[x^1-\beta x^0\right]}{\left[\left(\gamma\left[x^1-\beta x^0\right]\right)^2+\left(x^2\right)^2+\left(x^3\right)^2\right]^{\frac{3}{2}}} = \frac{E_1'}{c}$$

$$\frac{E_2}{c} = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{x^2}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{x^2}{\left[\left(\gamma\left[x^1-\beta x^0\right]\right)^2+\left(x^2\right)^2+\left(x^3\right)^2\right]^{\frac{3}{2}}} = \gamma\frac{E_2'}{c}$$

$$\frac{E_3}{c} = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{x^3}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{x^3}{\left[\left(\gamma\left[x^1-\beta x^0\right]\right)^2+\left(x^2\right)^2+\left(x^3\right)^2\right]^{\frac{3}{2}}} = \gamma\frac{E_3'}{c}$$

or, alternatively

$$\frac{\boldsymbol{E}}{c} = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{\boldsymbol{R}}{(R')^3},$$

where, as defined above, $\boldsymbol{R}$ is the vector from the current position $x^1=\beta x^0, x^2=0, x^3=0$ of the point charge to the point at which the field is evaluated.

The direction of $\boldsymbol{E}$ is illustrated in Figure 5. It can be noted that the $\boldsymbol{E}$ field is centred at the current position $x^1=\beta x^0, x^2=0, x^3=0$ of the point charge, rather than the retarded position. The magnitude of $\boldsymbol{E}$ can be expressed as

$$E = \frac{q}{4\pi\varepsilon_0 R^2}\frac{1}{\gamma^2}\left(\frac{1}{1-\beta^2\sin^2\theta}\right)^{\frac{3}{2}},$$

where θ is the angle between $\boldsymbol{R}$ and the direction of motion. The magnitude of $\boldsymbol{E}$ illustrated in Figure 6.

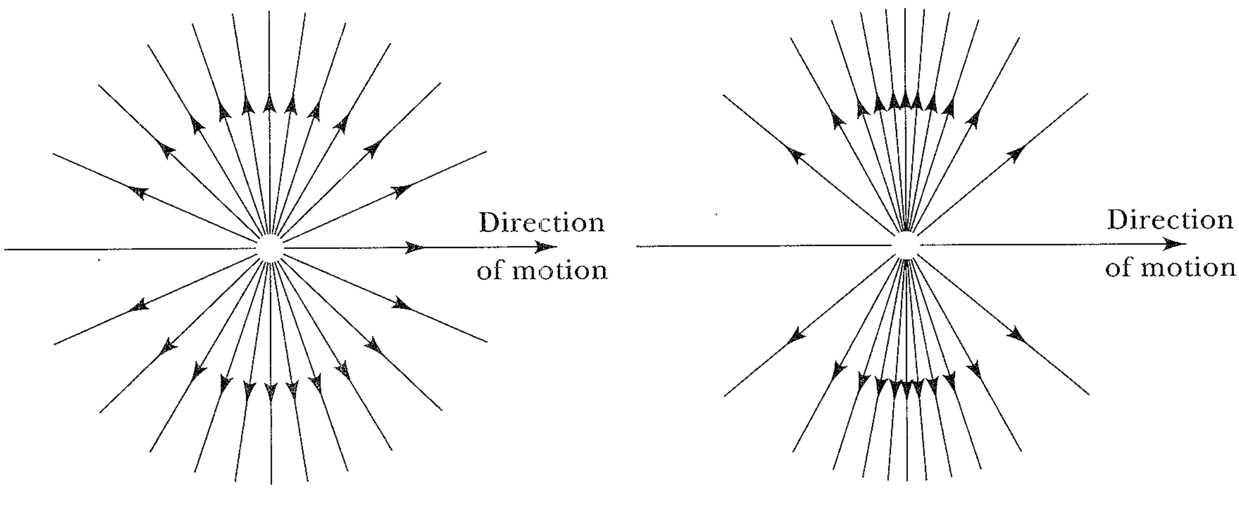

Figure 5: Electric field lines produced by a point charge moving with constant velocity for: $\beta = 0.7$ (left) and $\beta = 0.95$ (right) [1].

Developing our intuition:

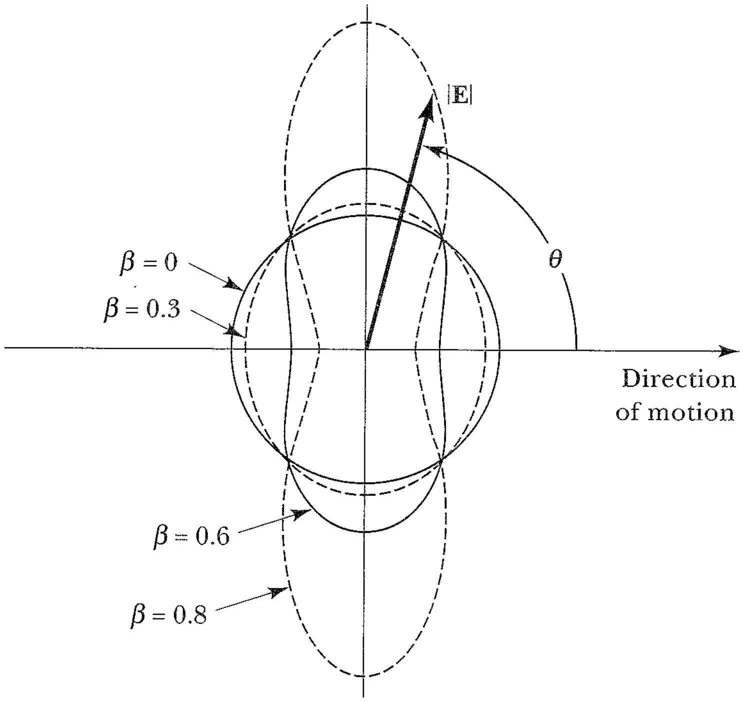

Figure 6: $|\boldsymbol{E}|$ at constant distance, R, from a moving point charge, shown as function of angle to direction of motion, θ, for different values of β [1].

1) It is again perhaps surprising that the field $\boldsymbol{E}$ at time x^0 is centred at the "current position" of the charge, $x^1 = \beta x^0$, rather than at the "retarded position" appropriate for a particular observation point? However, it is perhaps not surprising that this feature is common to both potentials and fields.

2) As a function of R (distance from the charge as measured in the frame in which it is moving) the magnitude $|\boldsymbol{E}|$ can be expressed as

$$E_1 = \frac{E_{1,0}}{\gamma^2}$$
$$E_\perp = \gamma E_{\perp,0}$$

where, in the directions parallel and perpendicular to the velocity, respectively, $E_{1,0}$ and $E_{\perp,0}$ are the $\boldsymbol{E}$ fields for a *stationary* particle at the current position $x^1 = \beta x^0, x^2 = 0, x^3 = 0$.

- The behaviour along the direction of motion relates to the Lorentz contraction of distances in that direction. The field gains a factor of $\frac{1}{\gamma^2}$ from its dependence on $\frac{1}{R^2}$, whereas at constant R' we would expect $E = E'$ for the component along the direction of motion.
- Distances transverse to the direction of motion are not Lorentz contracted and therefore $|\boldsymbol{E}|$ falls off with R in Figure 6 a factor of γ more slowly than it would for a stationary charge at the current position.

The corresponding B field is given by

$$
\begin{aligned}
B_1 &= 0 \\
B_2 &= -\beta\frac{E_3}{c} = -\frac{q}{4\pi\varepsilon_0 c}\gamma\beta\frac{x^3}{(R')^3} \\
B_3 &= \beta\frac{E_2}{c} = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{x^2}{(R')^3}
\end{aligned}
$$

or, alternatively

$$
\boldsymbol{B} = \frac{1}{c}\boldsymbol{\beta}\times\boldsymbol{E},
$$

and is illustrated in Figure 7.

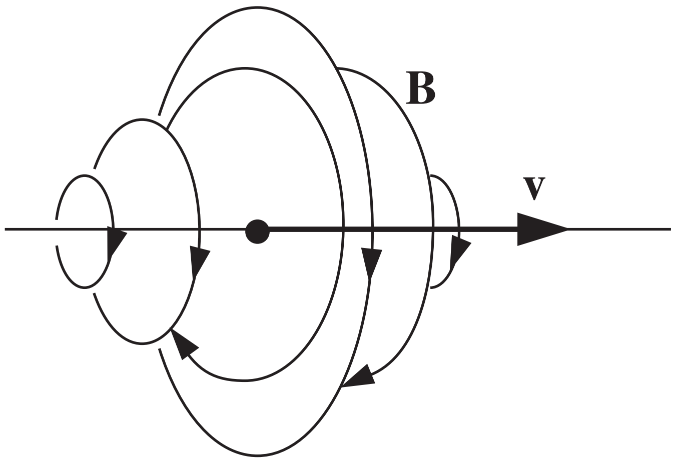

Figure 7: Lines of $\boldsymbol{B}$ relative to direction of motion [2].

5 Accelerating Charges: Potentials, Fields, and Radiation

The Liénard–Wiechert Potentials for a Moving Point Charge

It is important to note that only one space-time point on the world line of a point charge moving with speed $v < c$ can be connected to $P(\boldsymbol{r}, t)$ by a signal travelling at speed c.

The motion of the charge relative to the evaluation point modifies the potentials by a factor $\dfrac{1}{\left[1-\boldsymbol{\beta}\cdot\widehat{\boldsymbol{R}}\right]_{\text{ret}}}$ and gives the Liénard–Wiechert potentials

$$
V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left[R\left(1-\boldsymbol{\beta}\cdot\widehat{\boldsymbol{R}}\right)\right]_{\text{ret}}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{[R-\boldsymbol{\beta}\cdot\boldsymbol{R}]_{\text{ret}}} \qquad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\frac{qc\,[\boldsymbol{\beta}]_{\text{ret}}}{[R-\boldsymbol{\beta}\cdot\boldsymbol{R}]_{\text{ret}}} = \frac{[\boldsymbol{\beta}]_{\text{ret}}}{c}V(\boldsymbol{r},t)
$$

The subscript "ret" reminds us that that $\boldsymbol{R}$ and $\boldsymbol{\beta}$ are to be evaluated at the retarded time t_{ret}. (See Figure 8.)

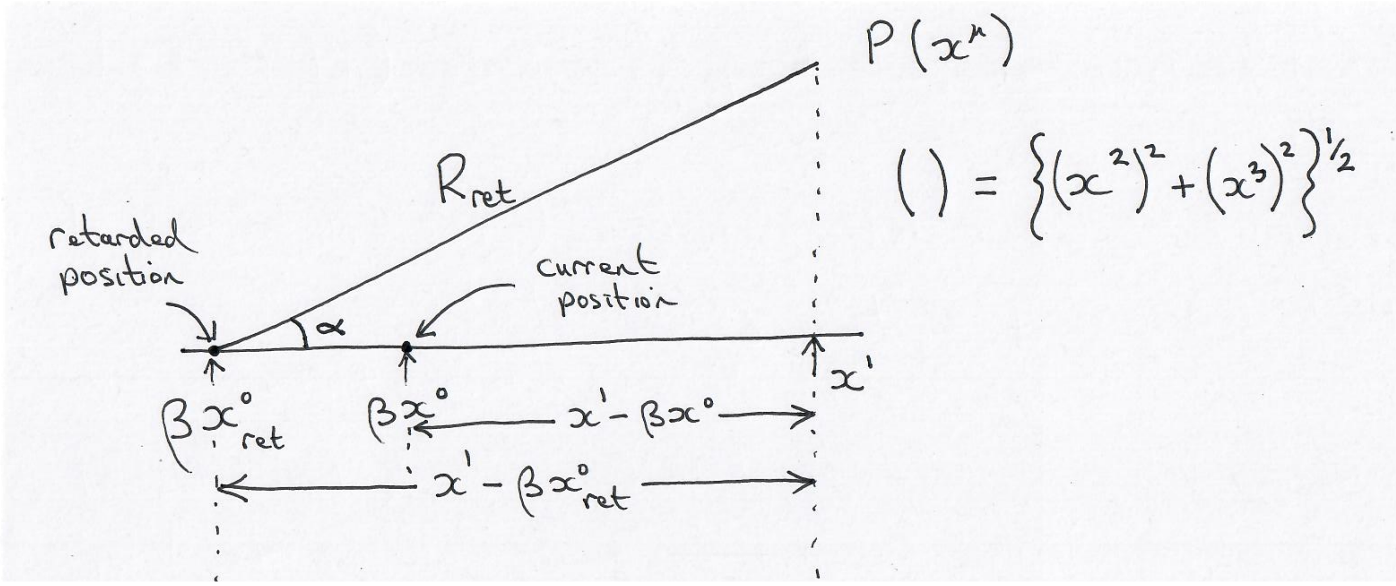

Figure 8: Sketch showing the relationship between $\boldsymbol{R}_{\rm ret}$ and $\boldsymbol{R}$ for a moving point charge.

The Liénard-Wiechert Fields for the Case that $\boldsymbol{\beta} \to 0$

The fields produced by a point charge q as its velocity $c\boldsymbol{\beta} \to 0$ are given by a simplified version of the *Liénard-Wiechert fields*

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}, t) &= \frac{q}{4\pi\epsilon_0}\left[\frac{\widehat{\boldsymbol{R}}}{R^2} + \frac{\widehat{\boldsymbol{R}} \times (\widehat{\boldsymbol{R}} \times \dot{\boldsymbol{\beta}})}{cR}\right]_{\rm ret} \\ \boldsymbol{B}(\boldsymbol{r}, t) &= \frac{1}{c}\left[\widehat{\boldsymbol{R}}\right]_{\rm ret} \times \boldsymbol{E}(\boldsymbol{r}, t),\end{aligned}$$

where $\dot{\boldsymbol{\beta}} \equiv \dfrac{\partial \boldsymbol{\beta}}{\partial t}_{\rm ret}$ is the time derivative of $\boldsymbol{\beta}$ (i.e., the acceleration divided by c).

The first term corresponds to the electrostatic potential for the point charge, but we are more interested here in the term proportional to $\dot{\boldsymbol{\beta}}$, which corresponds to radiation. A remarkable feature of this latter term is that the field strength depends on $\dfrac{1}{R}$ rather than $\dfrac{1}{R^2}$.

Note: The restriction $\beta \ll 1$ has an effect in various places in the derivations of the transverse components of the fields for an accelerating particle. For example,

- In Lecture 18 we assumed that the E field of the moving charged particle is spherically symmetric about the current position (no γ factors).
- In Lecture 19 we ignored the terms proportional to β when deriving the E field from the 4-potential.

The above has the effect that the expressions for the Liénard-Wiechert fields we obtained have no terms proportional to β.

(Not part of the core course) The full expressions for the Liénard-Wiechert fields and a sketch of their derivations are given in https://users.hep.manchester.ac.uk/test/twyatt/electrodynamics/LW-fields-handout.pdf.

Radiation from an Accelerating Point Charge

As found in two ways in the lectures, the transverse component of the electric field, $E_\perp$, produced by an accelerating point charge is given by:

$$E_\perp = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\sin\theta\,\dot{\beta}}{R}\right]_{\text{ret}}.$$

The Poynting vector may be written as

$$\mathbf{S} = \frac{1}{\mu_0 c}\left[E^2\widehat{\boldsymbol{R}} - (\widehat{\boldsymbol{R}}\cdot\boldsymbol{E})\boldsymbol{E}\right]_{\text{ret}} = \frac{1}{\mu_0 c}E^2\left[\widehat{\boldsymbol{R}}\right]_{\text{ret}},$$

since $\widehat{\boldsymbol{R}} \perp \boldsymbol{E}$.

The Poynting vector gives the radiated power per unit area. Since $|\boldsymbol{S}| \propto |\boldsymbol{E}\times\boldsymbol{B}| \propto \frac{1}{R^2}$. This naturally corresponds to radiation from the charge since the flux through a closed surface

$$\oint \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} \propto \oint \frac{1}{R}\frac{1}{R}R^2\,\mathrm{d}\Omega \to \text{a constant as } R\to\infty.$$

This may be contrasted with the case of constant velocity ($\dot{\boldsymbol{\beta}} = 0$). Since in this case $|\boldsymbol{E}| \propto \frac{1}{R^2}$ and $|\boldsymbol{B}| \propto \frac{1}{R^2}$, the Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \propto \frac{1}{R^4}$. This cannot correspond to radiation, since the flux through a closed surface

$$\oint \boldsymbol{S}\cdot \mathrm{d}\boldsymbol{a} \;\propto\; \oint \frac{1}{R^2}\frac{1}{R^2}R^2\,\mathrm{d}\Omega \;\to 0 \text{ as } R\to\infty.$$

For non-zero acceleration ($\dot{\boldsymbol{\beta}} \neq 0$) the following results hold

- For $\dot{\boldsymbol{\beta}} \neq 0$ the radiation term dominates as $R\to\infty$.
- The radiated power per unit solid angle is given by $\frac{\mathrm{d}P}{\mathrm{d}\Omega} = R^2 S$.
- The radiation is centred at the retarded position $\boldsymbol{r'}$ (i.e., the position of the point charge at the retarded time t_{ret}).

We consider radiation in three special cases

1) Non-relativistic, $\boldsymbol{\beta} \ll 1$:

The radiated power per unit solid angle is given by

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0 c q^2}{16\pi^2}\sin^2\theta\,\dot{\beta}^2,$$

where θ is the angle between $\dot{\boldsymbol{\beta}}$ and $\boldsymbol{R}$, as illustrated in Figure 9.

Integrating over all solid angles gives the total radiated power as

$$P = \frac{\mu_0 c q^2\dot{\beta}^2}{6\pi},$$

which is the *Larmor Formula*.

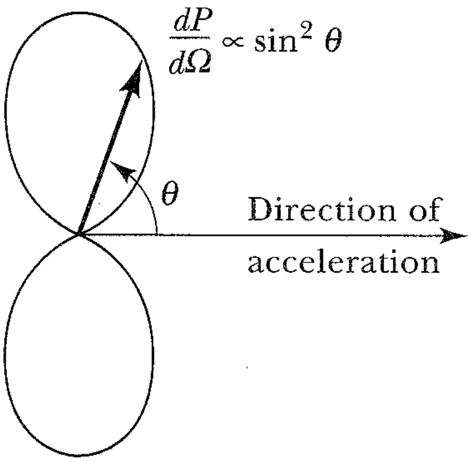

Figure 9: Angular dependence of (Larmor) radiation from a slowly moving accelerating charge $\beta \ll 1$ [1].

2) $\boldsymbol{\beta} > 0$, $\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta}$: "Bremsstrahlung Radiation"

For $\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta}$, the acceleration in the rest frame, $\dot{\beta}'$, and the acceleration in the frame in which the point charge is moving, $\dot{\beta}$, are related by $\dot{\beta}' = \gamma^3 \dot{\beta}$.

Transforming the total radiated power from the rest frame to the frame in which the point charge is moving gives

$$P_{\parallel} = \frac{\mu_0 c q^2 \dot{\beta}^2 \gamma^6}{6\pi}.$$

As derived in the lectures, the angular distribution of the radiated power in the rest frame, $\frac{\mathrm{d}P'}{\mathrm{d}\Omega'}$, and in the frame in which the point charge is moving, $\frac{\mathrm{d}P}{\mathrm{d}\Omega}$, are related by:

$$\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \gamma^4 \left(1 - \beta\cos\theta\right)^3 \frac{\mathrm{d}P}{\mathrm{d}\Omega}.$$

The radiated bremsstrahlung power per unit solid angle is then given by

$$\frac{\mathrm{d}P_{\parallel}}{\mathrm{d}\Omega} = \frac{\mu_0 c q^2}{16\pi^2} \frac{\sin^2\theta\, \dot{\beta}^2}{\left(1 - \beta\cos\theta\right)^5},$$

where θ is the angle between $\boldsymbol{\beta}$ and $\boldsymbol{R}$, as illustrated in Figure 10.

3) $\boldsymbol{\beta} > 0$, $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$: "Synchrotron Radiation"

For $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$, the acceleration in the rest frame, $\dot{\beta}'$, and the acceleration in the frame in which the point charge is moving, $\dot{\beta}$, are related by $\dot{\beta}' = \gamma^2 \dot{\beta}$.

Transforming the total radiated power from the rest frame to the frame in which the point charge is moving gives

$$P_{\perp} = \frac{\mu_0 c q^2 \dot{\beta}^2 \gamma^4}{6\pi}.$$

As illustrated in Figure 11, the radiated power per unit solid angle is given by

$$\frac{\mathrm{d}P_{\perp}}{\mathrm{d}\Omega} = \frac{\mu_0 c q^2 \dot{\beta}^2}{16\pi^2} \frac{1}{\left(1 - \beta\cos\theta\right)^3} \left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2 \left(1 - \beta\cos\theta\right)^2}\right).$$

where θ and ϕ are the polar and azimuthal angles, respectively, between $\boldsymbol{\beta}$ and $\boldsymbol{R}$.

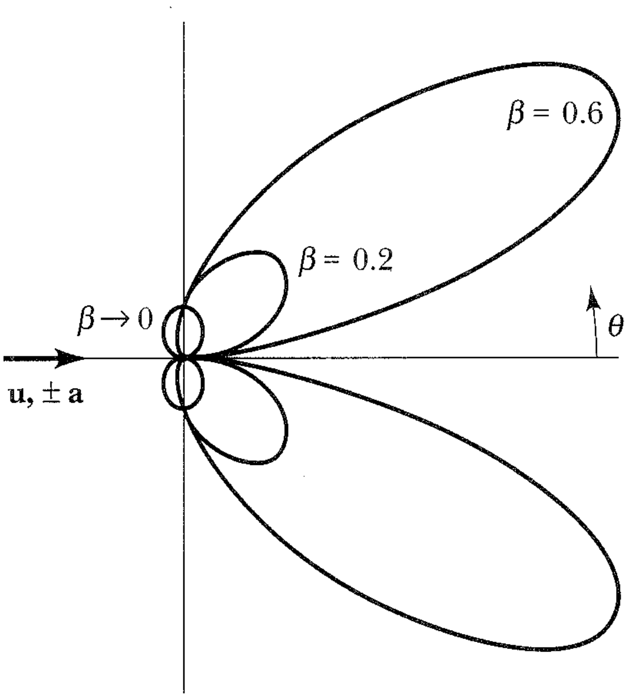

Figure 10: Angular dependence of bremsstrahlung radiation — the acceleration $\boldsymbol{a} = c\dot{\boldsymbol{\beta}}$ is parallel to the velocity $\boldsymbol{u} = c\boldsymbol{\beta}$ [1].

Multipole Radiation

Consider a pair of point charges of equal magnitude and opposite sign $\pm q$ centred at the origin and oscillating in position in anti-phase along the x-axis:

$$x_{\pm} = \pm x_o \cos\omega t, \qquad \ddot{x}_{\pm} = \mp\omega^2 x_o \cos\omega t.$$

This produces an oscillating electric dipole moment

$$\boldsymbol{p} = p_0 \cos(\omega t)\widehat{\boldsymbol{x}}, \text{ where } p_0 = 2qx_0.$$

We consider the following approximations:

$$\underbrace{x_0}_{\text{size of source}} \ll \underbrace{\lambda \sim \frac{c}{\omega}}_{\text{wavelength of radiation}} \ll \underbrace{r}_{\text{distance to source}}$$

The transverse components of the electric and magnetic fields produced by the two accelerating charges add coherently to give

$$E_T^{\text{dipole}} = 2E_T^{\text{singlecharge}} \qquad \text{andthus} \qquad P_{\text{dipole}} = 4P_{\text{singlecharge}}.$$

Taking an average over time gives the total average radiated power for an electric dipole as

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}.$$

Similarly, the radiated power per unit solid angle for an electric dipole is given by

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = r^2 \langle S \rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \sin^2\theta.$$

That is, the energy flow from radiation is radialy outwards and has the same “donut” angular distribution as for Larmor radiation, as given above.

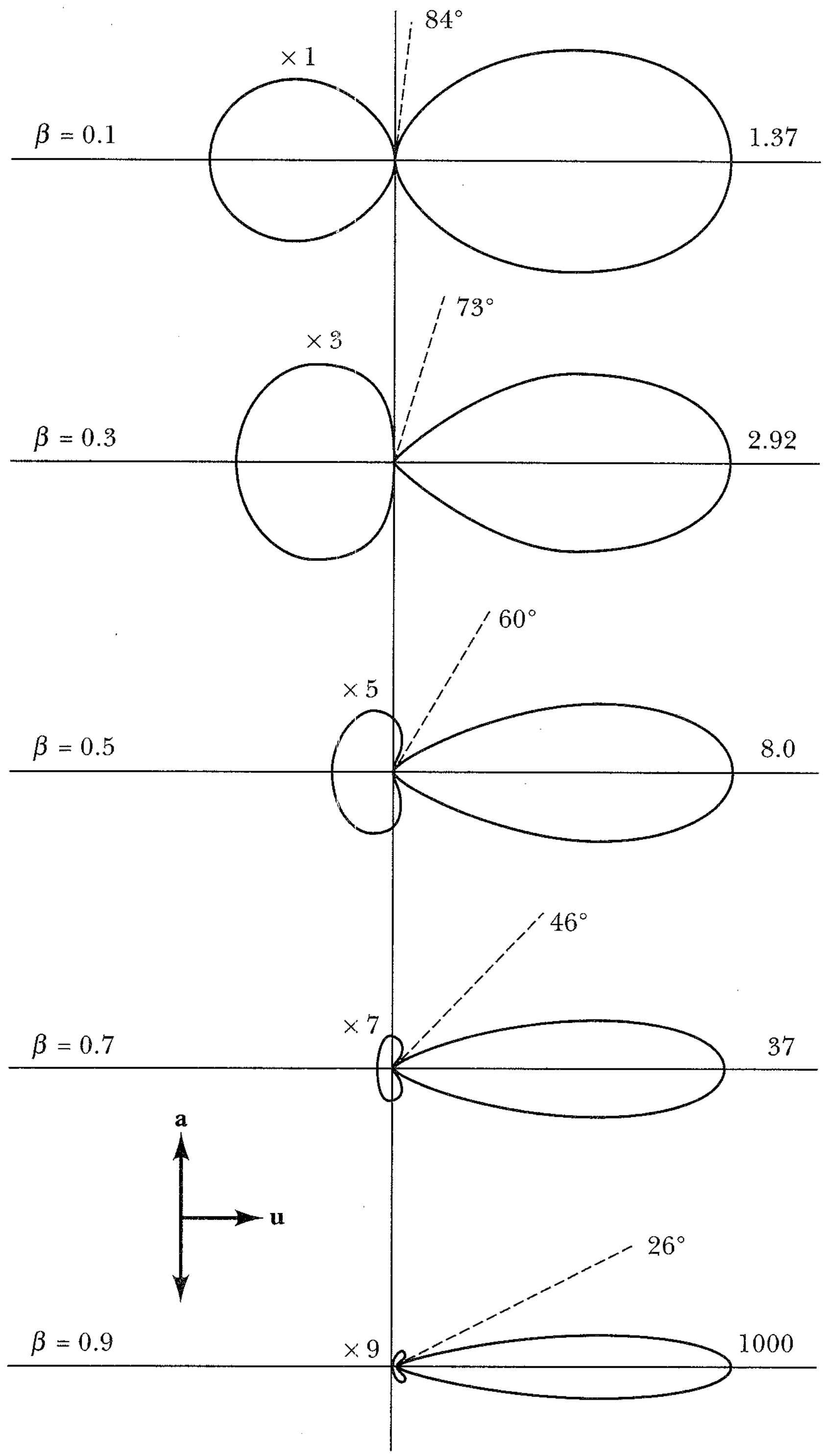

Figure 11: Angular dependence of synchrotron radiation in the plane of the orbit — the acceleration $\boldsymbol{a} = c\dot{\boldsymbol{\beta}}$ (vertical) is perpendicular to the velocity $\boldsymbol{u} = c\boldsymbol{\beta}$ (to the right) [1].

Scattering and the scattering cross section

The concept of a scattering cross section can be a bit difficult to get your head around the first time you meet it.

Let's start with the total cross section.

Let's imagine that an electron could be represented by a solid sphere that presents a certain cross sectional area to an incoming photon. If the photon passes within this cross sectional area, σ_T, the photon is scattered and otherwise the photon is not scattered.

For a single electron doing the scattering this would mean that for N incoming photons per unit area transverse to the direction of a beam, a number $N \times \sigma_T$ photons is scattered.

The same logic is used in Lecture 20 to calculate the fraction of the incoming power per unit area that is scattered.

Of course, in reality photons and electrons obey the laws of quantum mechanics and probability, but the classical description above is probably simpler to absorb at first.

The concept of a differential cross section, $\mathrm{d}\sigma/\mathrm{d}\Omega$, is even more abstract. It represents the part of the total cross section for which the photon would be scattered into a particular solid angle $\mathrm{d}\Omega$.

(And if you think cross sections are hard to get your head around, wait until you come across the concept of "luminosity" in particle physics, which has units of inverse area ;-)

6 Some Brief Notes on the Exam

Exam Contents

The overall format and general style of this year's exam will be similar to that of previous years (except for that of 2020–2021, which was affected by Covid-19). However, my precise choice of subjects to be covered is not identical to that of previous lecturers.

As you will have noticed already, the Electrodynamics course itself does involve quite a lot of "bookwork". My aim in the course has been to derive most of the important results from first principles. I hope that in your revision you will be able to focus on understanding deeply the material, as well as practising solving problems. The derivations I elected to work through in the lectures were chosen to provide important physical insights into the workings of Electrodynamics, rather than being mere exercises in algebra. If you can sit down with a blank piece of paper and work through for yourself the main derivations, that would be a good test of your understanding of the physics.

You should consider working conscientiously[3] through all the Example Sheets I have provided as an *absolute bare minimum* of practice in problem solving that you need to prepare yourself for the exam in Electrodynamics. I also strongly recommend that you work through the past few years' Electrodynamics exams, which are linked from the course web page. A number of our Wednesday sessions were also devoted to working through illustrative past exam questions. Large numbers of additional problems are available in text books such as those by Heald and Marion [1], Griffiths [2], and Jackson [3], and more generally on the web. I have provided some useful links to additional material on the course web page.

A page of potentially useful formulae will be provided at the beginning of the exam. This page is given as an Appendix to this course summary.

Here are a few specific technical recommendations regarding using the index notation in special relativity. In the lectures and notes I have tried to be very careful to distinguish between, e.g.,

- x^2 (the magnitude of the 2nd spatial component of the 4-vector x^μ),
- $(x)^2$ $(= x^\mu x_\mu$, the square of the 4-vector), and
- $(x^2)^2$ (the square of the magnitude of the 2nd spatial component of the 4-vector x^μ).

I strongly recommend you follow the same conventions when writing answers to questions in order to avoid any ambiguity.

The approximate balance between bookwork and solving problems in January's exam will be fairly similar to that of recent years' exams (excepting 2020–2021). Here is a list of some specific derivations/proofs I shall **not** expect you to reproduce in the exam.

- Derivation of the expansion of $\dfrac{1}{R}$ in powers of $\dfrac{r'}{r}$ and Legendre Polynomials (in Lecture 5).
- Formal proof that $\boldsymbol{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\boldsymbol{j}\,(\boldsymbol{r'})}{R}\,\mathrm{d}\tau'$ is consistent with $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ (in Lecture 7).
- Formal proof that $V\,(\boldsymbol{r}, t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \dfrac{\rho\,(\boldsymbol{r'}, t')}{R}\,\mathrm{d}\tau'$ is a solution to the wave equation (in Lecture 9).

[3] "Working conscientiously" obviously means trying very hard to solve the problems yourself. Just looking at my answers without trying hard yourself will bring few benefits.

I shall **not** expect you to **remember** the equations that define the Liénard-Wiechert **fields** in the rest frame, or to **derive** them from the Liénard-Wiechert potentials (as discussed in Lecture 19). However, you may be given them in the exam script and be expected to **use** them to prove any of the results for accelerating point particles given in the remainder of section 5 above. I would consider it legitimate to set a question that involved the translation between the various formulations of the *potentials* produced by a point particle (e.g., the rest and moving frame formulations in Lecture 14 and the alternative Liénard-Wiechert potentials, as discussed in Lecture 19).

Past Exams

Past exam papers can be a useful source of questions to help you test your understanding of the course material and practice your problem solving skills. However, you may need to take into account that some topics included by previous lecturers may not have been covered this year and vice versa.

One important change in the way I teach radiation from accelerating charged particles compared to the approach used by previous lecturers in Manchester is that I neither state the full the Liénard-Wiechert fields, nor use them to derive the expressions for radiation from accelerating charges. The approach I have taken instead is to derive the Liénard-Wiechert fields for the special case that $\beta \ll 1$, to use these to derive the radiation in the rest frame, and then to use the Lorentz transformations to derive the expressions for Bremsstrahlung and Synchrotron radiation. Personally, I think this new approach provides a much better insight into the underlying physics, and it introduces general techniques that are very useful for particle physics and high energy astrophysics. This change to the course does mean that some of the "bookwork" items at the beginning of some past exam questions on radiation set by previous lecturers might look a little obscure. However, this year's lectures should still have given you enough understanding to work out the answers with a little thought, if you are interested.

When looking at past exam papers from January 2013 to January 2018, inclusive, you need to be aware of one very important technical difference between my lectures and those given by the previous lecturer. I have used the metric tensor

$$g^{\mu\nu} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

which is that used, e.g., in the text book by Jackson [3] and in the 1st year Advanced Dynamics course.

The previous lecturer used the metric tensor

$$g^{\mu\nu} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is that used, e.g., in the text book by Griffiths [2].

This has many knock on effects if you look at past exam papers. These are all "trivial", but can cause confusion if you are not careful. For example, the sign of all 4-vector products is inverted. The d'Alembertian changes sign, and so the wave equation becomes $\partial^\mu \partial_\mu A^\mu = \Box^2 A^\mu = -\mu_0 j^\mu$. All elements of the field tensor are multiplied by -1, etc.

References

[1] *'Classical Electromagnetic Radiation (3rd edition)'*, M.A.Heald and J.B.Marion, Saunders College Publishing.

[2] *'Introduction to Electrodynamics (4th edition)'*, D.J.Griffiths, Pearson Education.

[3] *'Classical Electrodynamics (3rd edition)'*, J.D.Jackson, John Wiley and Sons, Inc.

Appendix: Formula Page to be Provided on the Exam Paper

The first three **Legendre Polynomials** are given by

$$P_0(\cos\theta) = 1, \quad P_1(\cos\theta) = \cos\theta, \quad P_2(\cos\theta) = (3\cos^2\theta - 1)/2.$$

Vector Calculus

$$\begin{aligned}
\boldsymbol{\nabla}(\phi\psi) &= \phi\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\phi, \\
\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v}) + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{u}) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u}, \\
\boldsymbol{\nabla}\cdot(\phi\mathbf{u}) &= \phi(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mathbf{u}\cdot(\boldsymbol{\nabla}\phi), \\
\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) &= \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u}) - \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v}), \\
\boldsymbol{\nabla}\times(\phi\mathbf{u}) &= \phi(\boldsymbol{\nabla}\times\mathbf{u}) - \mathbf{u}\times(\boldsymbol{\nabla}\phi), \\
\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{v}) &= (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} + \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{u})), \\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \boldsymbol{\nabla}^2\mathbf{u}.
\end{aligned}$$

Spherical Polar Coordinates

$$\begin{aligned}
\boldsymbol{\nabla} f &= \frac{\partial f}{\partial r}\widehat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\widehat{\boldsymbol{\phi}}, \\
\boldsymbol{\nabla}\cdot\boldsymbol{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \\
\boldsymbol{\nabla}\times\boldsymbol{F} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right]\widehat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\widehat{\boldsymbol{\theta}} \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]\widehat{\boldsymbol{\phi}}.
\end{aligned}$$

Cylindrical Coordinates

$$\begin{aligned}
\boldsymbol{\nabla} f &= \frac{\partial f}{\partial s}\widehat{\boldsymbol{s}} + \frac{1}{s}\frac{\partial f}{\partial\phi}\widehat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\widehat{\boldsymbol{z}}, \\
\boldsymbol{\nabla}\cdot\boldsymbol{F} &= \frac{1}{s}\frac{\partial}{\partial s}(sF_s) + \frac{1}{s}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z} \\
\boldsymbol{\nabla}\times\boldsymbol{F} &= \left[\frac{1}{s}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right]\widehat{\boldsymbol{s}} + \left[\frac{\partial F_s}{\partial z} - \frac{\partial F_z}{\partial s}\right]\widehat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sF_\phi) - \frac{\partial F_s}{\partial\phi}\right]\widehat{\boldsymbol{z}}.
\end{aligned}$$

Matrix Representations for Relativity

Metric tensor — Lorentz boost along x^1 axis — Electromagnetic field tensor

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F^{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_1}{c} & -\frac{E_2}{c} & -\frac{E_3}{c} \\ \frac{E_1}{c} & 0 & -B_3 & B_2 \\ \frac{E_2}{c} & B_3 & 0 & -B_1 \\ \frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix}$$

Radiation from a non-relativistic particle

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0 c q^2}{16\pi^2}\sin^2\theta\,\dot{\beta}^2, \qquad P = \frac{\mu_0 c q^2 \dot{\beta}^2}{6\pi}.$$