

EXAM QUESTION JANUARY 2011

PHYS30441

1. Potential and a spherical shell

A conducting sphere of radius a , at a potential V_0 , is surrounded by a thin concentric shell of charge. The radius of the shell is b and the surface charge density is:

$$\sigma(\theta) = k \cos \theta$$

where k is a constant and θ refers to spherical coordinates.

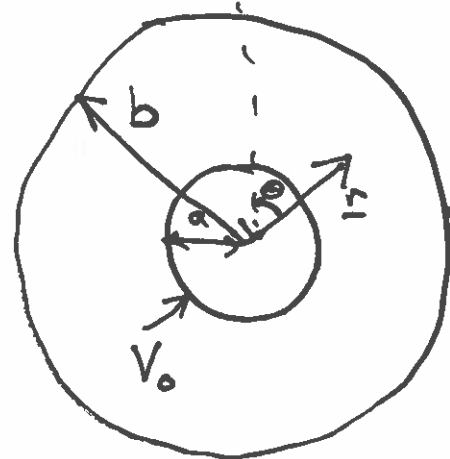
- a) What are the boundary conditions on the scalar potential V and electric field $\vec{E}$ at $r = b$? [3 marks]
- b) What is the electric field within the conductor ($r < a$)? [2 marks]
- c) Find the electric potential inside ($a < r < b$) and outside ($r > b$) the surrounding shell by considering axially symmetric solutions to the Laplace equation, $\nabla^2 V = 0$. You should consider solutions of the form:

$$V(r, \theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta)$$

and you may find the following Legendre functions helpful in your analysis: $P_0(\cos \theta) = 1$, $P_1(\cos \theta) = \cos \theta$. Ensure that you clearly indicate all boundary conditions that are applied in your answer. [14 marks]

- d) Find the surface charge $\sigma_r(\theta)$ on the conductor ($r=a$). [4 marks]
- e) What is the total charge on the conductor ($r=a$)? [2 marks]

P.T.O.



$$a < r < b : V_1(r, \theta)$$

$$r > b : V_2(r, \theta)$$

$$\sigma(\theta) = k \cos \theta$$

(a) boundary conditions at $r = b$?

$$V \text{ continuous : } V_1(b, \theta) = V_2(b, \theta)$$

$$E_{\text{radial}}^{\text{outside}} - E_{\text{radial}}^{\text{inside}} = -\left. \frac{\partial V_2}{\partial r} \right|_{r=b} + \left. \frac{\partial V_1}{\partial r} \right|_{r=b} = \frac{\sigma(\theta)}{\epsilon_0}$$

$$(b) \quad E = 0 \quad \text{for} \quad r < a$$

Other boundary conditions?

$$V_2 \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

N.B. V_1 as $r \rightarrow \infty$ is irrelevant because
 V_1 doesn't apply in that region!

(c) Consider region $a < r < b$:

$$V_i(r, \theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos \theta)$$

Boundary conditions $r = a$

$$V_i(a, \theta) = V_0$$

$$V_0 = A_0 + \frac{B_0}{a}$$

$$0 = A_n a^{2n+1} + B_n \quad \text{for } n \neq 0$$

$$\therefore V_i(r, \theta) = \frac{aV_0}{r} + \sum_{n=0}^{\infty} A_n \left(r^n - \frac{a^{2n+1}}{r^{n+1}} \right) P_n(\cos \theta) \quad (A)$$

Explanation added in answer to question after lecture:

$$B_0 = aV_0 - aA_0$$

This part is present only for $n=0$ and so is taken out of the general sum.

This part is accounted for here in the general sum.

Cross-check plug $r = a$ into (A)!

Consider the region $r > b$:

$$V_2(r, \theta) = \sum_{n=0}^{\infty} \left(\cancel{C_n} r^n + \frac{D_n}{r^{n+1}} \right) P_n(\cos \theta)$$

Boundary condition: $V_2 \rightarrow 0$ as $r \rightarrow \infty$ $C_n = 0 \quad \forall n$

$V_1(b, \theta) = V_2(b, \theta)$ and equate coefficient of each P_n

$$\frac{aV_0}{b} + A_0 \left(1 - \frac{a}{b} \right) = \frac{D_0}{b}$$

$$A_n \left(b^n - \frac{a^{2n+1}}{b^{n+1}} \right) = \frac{D_n}{b^{n+1}} \quad n \neq 0$$

$$V_2(r, \theta) = \frac{aV_0}{r} + \sum_{n=0}^{\infty} A_n \left(\frac{b^{2n+1} - a^{2n+1}}{r^{n+1}} \right) P_n(\cos \theta) \quad (B)$$

Boundary condition on E_{radial} at $r=b$ using (A) & (B)

$$\frac{\sigma(\theta)}{\epsilon_0} = \frac{k \cos \theta}{\epsilon_0} = \frac{k P_1(\cos \theta)}{\epsilon_0} = -\frac{\partial V_2}{\partial r} + \frac{\partial V_1}{\partial r}$$

Exercise for you!

$$= \sum_{n=0}^{\infty} A_n (2n+1) b^{n-1} P_n(\cos \theta)$$

$$\therefore 3A_1 = \frac{k}{\epsilon_0} \quad \& \quad A_n = 0 \quad \text{for } n \neq 1.$$

From (A) : $V_1(r, \theta) = \frac{aV_0}{r} + \frac{k}{3\epsilon_0} \left(r - \frac{a^3}{r^2} \right) \cos \theta$

" (B) : $V_2(r, \theta) = \frac{aV_0}{r} + \frac{k}{3\epsilon_0} \left(\frac{b^3 - a^3}{r^2} \right) \cos \theta$

Cross-check : Do these expressions satisfy the
boundary conditions

Exercise for you Do parts (d) and (e) of the question!