EXAM QUESTION JANUARY 2011

1. Potential and a spherical shell

A conducting sphere of radius a, at a potential V_0, is surrounded by a thin concentric shell of charge. The radius of the shell is b and the surface charge density is:

$$\sigma(\theta) = k\cos\theta$$

where k is a constant and θ refers to spherical coordinates.

a) What are the boundary conditions on the scalar potential V and electric field $\vec{E}$ at $r = b$? [3 marks]

b) What is the electric field within the conductor $(r < a)$? [2 marks]

c) Find the electric potential inside $(a < r < b)$ and outside $(r > b)$ the surrounding shell by considering axially symmetric solutions to the Laplace equation, $\nabla^2 V = 0$. You should consider solutions of the form:

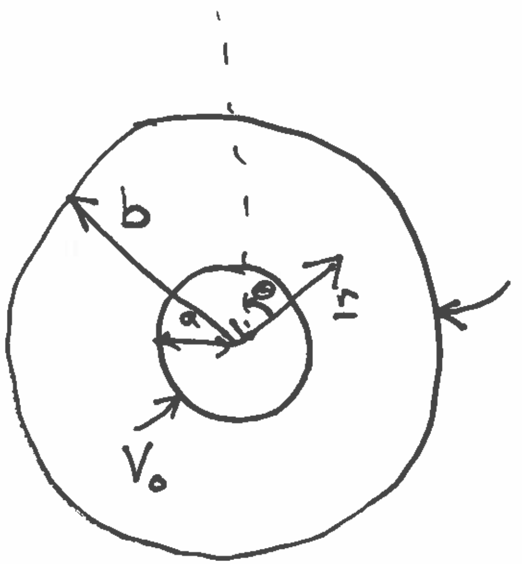
$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

and you may find the following Legendre functions helpful in your analysis: $P_0(\cos\theta) = 1, P_1(\cos\theta) = \cos\theta$. Ensure that you clearly indicate all boundary conditions that are applied in your answer. [14 marks]

d) Find the surface charge $\sigma_i(\theta)$ on the conductor ($r=a$). [4 marks]

e) What is the total charge on the conductor ($r=a$)? [2 marks]

P.T.O.



$a < r < b$: $V_1(r,\theta)$

$r > b$: $V_2(r,\theta)$

$\sigma(\theta) = k\cos\theta$

(a) boundary conditions at $r = b$?

V continuous : $V_1(b,\theta) = V_2(b,\theta)$

$$E^{\text{outside}}_{\text{radial}} - E^{\text{inside}}_{\text{radial}} = -\frac{\partial V_2}{\partial r}\bigg|_{r=b} + \frac{\partial V_1}{\partial r}\bigg|_{r=b} = \frac{\sigma(\theta)}{\varepsilon_0}$$

(b) $E = 0$ for $r < a$

Other boundary conditions?

$$V_2 \to 0 \quad \text{as} \quad r \to \infty$$

N.B. V_1 as $r \to \infty$ is irrelevant because V_1 doesn't apply in that region!

(c) Consider region $a < r < b$:

$$V_1(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

Boundary conditions $r = a$

$$V_1(a,\theta) = V_0$$

$$\therefore \quad V_0 = A_0 + \frac{B_0}{a}$$

$$0 = A_n a^{2n+1} + B_n \quad \text{for } n \neq 0$$

Explanation added in answer to question after lecture:

$B_0 = aV_0 - aA_0$

This part is present only for $n=0$ and so is taken out of the general sum.

This part is accounted for here in the general sum.

$$\therefore \quad V_1(r,\theta) = \frac{aV_0}{r} + \sum_{n=0}^{\infty} A_n \left(r^n - \frac{a^{2n+1}}{r^{n+1}} \right) P_n(\cos\theta) \qquad (A)$$

Cross-check plug $r = a$ into (A) !

Consider the region $r > b$:

$$V_2(r,\theta) = \sum_{n=0}^{\infty} \left(\cancel{C_n} r^n + \frac{D_n}{r^{n+1}} \right) P_n(\cos\theta)$$

Boundary condition: $V_2 \to 0$ as $r \to \infty$ $\quad C_n = 0 \quad \forall n$

$V_1(b,\theta) = V_2(b,\theta)$ and equate coefficient of each P_n

$$\frac{aV_0}{b} + A_0\left(1 - \frac{a}{b}\right) = \frac{D_0}{b}$$

$$A_n\left(b^n - \frac{a^{2n+1}}{b^{n+1}}\right) = \frac{D_n}{b^{n+1}} \qquad n = \emptyset$$

$$V_2(r,\theta) = \frac{aV_0}{r} + \sum_{n=0}^{\infty} A_n \left(\frac{b^{2n+1} - a^{2n+1}}{r^{n+1}} \right) P_n(\cos\theta) \quad (B)$$

Cross check!

Boundary condition on E_{radial} at $r = b$ using (A) & (B)

$$\frac{\sigma(\theta)}{\varepsilon_0} = \frac{k\cos\theta}{\varepsilon_0} = \frac{k P_1(\cos\theta)}{\varepsilon_0} = -\frac{\partial V_2}{\partial r} + \frac{\partial V_1}{\partial r}$$

Exercise for you!

$$= \sum_{n=0}^{\infty} A_n (2n+1) b^{n-1} P_n(\cos\theta)$$

$$\therefore 3A_1 = \frac{k}{\varepsilon_0} \quad \& \quad A_n = 0 \text{ for } n \neq 1.$$

Please see completed answers and notes on following pages.

Boundary condition on E_r at $r=b$ (using eqns. A & B):

$$\frac{\sigma(\theta)}{\varepsilon_0} = \frac{k\cos\theta}{\varepsilon_0} = \frac{kP_1}{\varepsilon_0} = -\left.\frac{\partial V_2}{\partial r}\right|_{r=b} + \left.\frac{\partial V_1}{\partial r}\right|_{r=b}$$

$$= -\left[-\frac{aV_0}{b^2} + \sum_{n=0}^{\infty} -(n+1)A_n \frac{\left(b^{2n+1} - a^{2n+1}\right)}{b^{n+2}} P_n(\cos\theta) \right]$$

$$+ \left[-\frac{aV_0}{b^2} + \sum_{n=0}^{\infty} A_n \left(nb^{n-1} - a^{2n+1}\frac{(-[n+1])}{b^{n+2}} \right) P_n(\cos\theta) \right]$$

$$= \sum_{n=0}^{\infty} A_n (2n+1)\left(b^{n-1}\right) P_n(\cos\theta)$$

Equating coefficients for each P_n

$$3A_1 = \frac{k}{\cancel{3}\varepsilon_0}$$

$$A_n = 0 \quad \text{for} \quad n \neq 1$$

From (A) : $$V_1(r,\theta) = \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

" (B) : $$V_2(r,\theta) = \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(\frac{b^3 - a^3}{r^2}\right)\cos\theta$$

<u>Cross-check</u>: Do these expressions satisfy the boundary conditions

<u>Exercise for you</u> Do parts (d) and (e) of the question!

$$V_1(r,\theta) = \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

$$V_2(r,\theta) = \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(\frac{b^3 - a^3}{r^2}\right)\cos\theta$$

(d) **B**oundary condition at $r=a$

$$\frac{\sigma_1(\theta)}{\varepsilon_0} = -\left.\frac{\partial V_1}{\partial r}\right|_{r=a} = -\left[-\frac{aV_0}{a^2} + \frac{k}{3\varepsilon_0}\left(1-(-2)\frac{a^3}{a^3}\right)\cos\theta\right]$$

$$= \frac{V_0}{a} - \frac{k\cos\theta}{\varepsilon_0}$$

[Note -ve sign for $k\cos\theta$ term as we guessed at the start!]

$$\sigma_1(\theta) = \frac{\varepsilon_0 V_0}{a} - k\cos\theta$$

(e) Total charge on conductor

but do we need to ?

$$Q_{total} = \int_0^{2\pi} d\phi \int_0^{\pi} a^2\sin\theta\,\sigma_1(\theta)\,d\theta = \underbrace{4\pi a^2}_{\text{area of sphere}}\frac{\varepsilon_0 V_0}{a} = 4\pi\varepsilon_0 a V_0\,,$$

since $-k\cos\theta$ integrates to zero over interval $0<\theta<\pi$.

Can anyone see a clever way of just writing down this answer from $V_1(r,\theta)$ without going through the intermediate stage of calculating $\sigma_1(\cos\theta)$!?

Monopole term in potential $V_{monopole} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{total}}{r}$

$\therefore \quad \frac{a V_0}{r} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{total}}{r}$ gives the same result!

Let's review what we did

1) Drew diagram and thought about the physics

2) Worked out the boundary conditions

3) Wrote general solutions and applied boundary conditions

4) Used some physics insight to save us the trouble of doing a boring integration.

Note added in response to a question after the lecture

When considering boundary conditions we can apply only those relevant to the region of validity for a given potential V.

- E.g. in this problem the b.c. that $V \to 0$ as $r \to \infty$ can be applied only to V_2, but not V_1, because V_1 is valid only for the region $a < r < b$.
- In problems where $V(r, \cos\theta)$ has to be found in the region that includes the origin ($r = 0$) the requirement that V is finite $\forall r$ allows us to set to zero the coefficients of all terms in $\frac{1}{r^{n+1}}$.

  However, in this problem neither V_1 nor V_2 are valid for $r < a$, and so their behaviour as $r \to 0$ is not relevant.