Electrodynamics (PHYS30441) Exercises: Revision of vector calculus Terry Wyatt

These exercises are designed to refresh your memory of vector calculus and to prepare you for some of the derivations you will meet later in the course.

1. In this problem $\phi \equiv \phi(\mathbf{r})$ is a scalar function of position and $\mathbf{u} \equiv \mathbf{u}(\mathbf{r})$ and $\mathbf{v} \equiv \mathbf{v}(\mathbf{r})$ are vector fields. Use the Cartesian forms of $\mathbf{u} \times \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$, $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\times$ and $\boldsymbol{\nabla}\cdot$ to demonstrate the vector identities given below.

   I suggest you do this in two ways:

   (a) explicitly considering the x component of the vector equations;

   (b) using general index notation for vectors.

   Remember that once these are proved in Cartesian co-ordinates they will be automatically valid in *any* co-ordinate system.

   (a) $\nabla \times (\phi \boldsymbol{u}) = \phi \nabla \times \boldsymbol{u} - \boldsymbol{u} \times (\nabla \phi)$,

   (b) $\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v}.\nabla)\boldsymbol{u} - (\boldsymbol{u}.\nabla)\boldsymbol{v} + \boldsymbol{u}(\nabla.\boldsymbol{v}) - \boldsymbol{v}(\nabla.\boldsymbol{u})$

   (c) $\frac{1}{2}\nabla(\boldsymbol{u}.\boldsymbol{u}) = \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) + (\boldsymbol{u}.\nabla)\boldsymbol{u}$,

   (d) $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla.\boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.

   Note: $(\boldsymbol{u}.\nabla)\boldsymbol{v} = (u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z})(v_x \hat{\boldsymbol{x}} + v_y \hat{\boldsymbol{y}} + v_z \hat{\boldsymbol{z}})$

2. In this question $\boldsymbol{a}$ is a constant vector which does not depend on $\boldsymbol{r}$. Prove the following expressions using the vector identities from question 1 where appropriate:

   (a) $\nabla.\boldsymbol{r} = 3$,

   (b) $\nabla(r^n) = nr^{n-2}\boldsymbol{r}$, (In particular, note $\nabla\left(\frac{1}{r}\right) = -\frac{\boldsymbol{r}}{r^3} = -\frac{1}{r^2}\hat{\boldsymbol{r}}$)

   (c) $\nabla \times (r^n \boldsymbol{a}) = nr^{n-2}\boldsymbol{r} \times \boldsymbol{a}$,

   (d) $(\boldsymbol{a}.\nabla)\boldsymbol{r} = \boldsymbol{a}$,

   (e) $\nabla \times (\boldsymbol{a} \times \boldsymbol{r}) = 2\boldsymbol{a}$,

   (f) $\nabla(r^n \boldsymbol{a}.\boldsymbol{r}) = r^{n-2}[r^2\boldsymbol{a} + n(\boldsymbol{a}.\boldsymbol{r})\boldsymbol{r}]$,

   (g) $\nabla \times (r^n \boldsymbol{a} \times \boldsymbol{r}) = r^{n-2}[(n+2)r^2\boldsymbol{a} - n(\boldsymbol{a}.\boldsymbol{r})\boldsymbol{r}]$.