PHYS30441

Electrodynamics

Exercises: Revision of vector calculus

Solutions - Terry Wyatt.

(a) Demonstrations for $\hat{\underline{x}}$ component

Since in catesian coordinates the three coordinates can be treated on an equal footing:

if we can prove the required relationships hold for one coordinate then they are guaranteed to hold for the other two, and therefore to hold in general.

Just for definiteness we'll look at the $\hat{\underline{x}}$ component

Q1 (a)

$$\nabla \times (\phi \underline{u}) = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi u_x & \phi u_y & \phi u_z \end{vmatrix}$$

$\hat{\underline{x}}$ component of $\nabla \times (\phi \underline{u})$

$$= \frac{\partial}{\partial y}(\phi u_z) - \frac{\partial}{\partial z}(\phi u_y)$$

$$= \left\{ \frac{\partial \phi}{\partial y} u_z - \frac{\partial \phi}{\partial z} u_y \right\} + \phi \left\{ \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right\}$$

$$= \left[(\nabla \phi) \times \underline{u} + \phi (\nabla \times \underline{u}) \right]_{\hat{x} \text{ component}}$$

$$= \left[\phi (\nabla \times \underline{u}) - \underline{u} \times (\nabla \phi) \right]_{\hat{\underline{x}} \text{ component}}$$

$$\therefore \quad \nabla \times (\phi \underline{u}) = \phi (\nabla \times \underline{u}) - \underline{u} \times (\nabla \phi)$$

Q1(b) $\hat{\underline{x}}$ component of $\nabla \times (\underline{u} \times \underline{v})$

$$= \frac{\partial}{\partial y}\left[\underline{u} \times \underline{v}\right]_{\hat{\underline{z}}} - \frac{\partial}{\partial z}\left[\underline{u} \times \underline{v}\right]_{\hat{\underline{y}}}$$

$$= \frac{\partial}{\partial y}\left(u_x v_y - u_y v_x\right) - \frac{\partial}{\partial z}\left(u_z v_x - u_x v_z\right)$$

$$= v_y \frac{\partial u_x}{\partial y} + u_x \frac{\partial v_y}{\partial y} - v_x \frac{\partial u_y}{\partial y} - u_y \frac{\partial v_x}{\partial y}$$

$$- v_x \frac{\partial u_z}{\partial z} - u_z \frac{\partial v_x}{\partial z} + v_z \frac{\partial u_x}{\partial z} + u_x \frac{\partial v_z}{\partial z}$$

$$= \left(\quad v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) u_x - \left(\quad u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) v_x$$

$$+ \left(\quad \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) u_x - \left(\quad \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) v_x$$

Adding and subtracting $v_x \frac{\partial u_x}{\partial x}$ to the first and fourth terms and $u_x \frac{\partial v_x}{\partial x}$ to the second and third terms:

$$= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right) u_x - \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) v_x$$

$$+ \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) u_x - \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) v_x$$

$$= \left[(\underline{v} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{v} + (\nabla \cdot \underline{v})\underline{u} - (\nabla \cdot \underline{u})\underline{v} \right]$$

$\hat{\underline{x}}$ component.

which proves the general result.

Q1 (c) $\hat{\underline{x}}$ component of $\frac{1}{2}\nabla(\underline{u} \cdot \underline{u})$

$$= \frac{1}{2}\frac{\partial}{\partial x}\left(u_x^2 + u_y^2 + u_z^2\right)$$

$$= u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_y}{\partial x} + u_z \frac{\partial u_z}{\partial x}$$

$\hat{\underline{x}}$ component of $(\underline{u} \cdot \nabla)\underline{u}$

$$= u_x \frac{\partial u_x}{\partial x} + \cancel{u_y \frac{\partial u_x}{\partial y}} + \cancel{u_z \frac{\partial u_x}{\partial z}}$$

$\hat{\underline{x}}$ component of $\underline{u} \times (\nabla \times \underline{u})$

$$= u_y \left[\nabla \times \underline{u}\right]_z - u_z \left[\nabla \times \underline{u}\right]_y$$

$$= u_y \frac{\partial u_y}{\partial x} - \cancel{u_y \frac{\partial u_x}{\partial y}} - \cancel{u_z \frac{\partial u_x}{\partial z}} + u_z \frac{\partial u_z}{\partial x}$$

This demonstrates that for the $\hat{\underline{x}}$ components

$$\frac{1}{2}\nabla(\underline{u} \cdot \underline{u}) = \underline{u} \times (\nabla \times \underline{u}) + (\underline{u} \cdot \nabla)\underline{u}$$

Q1(d) $\hat{\underline{x}}$ component of $\nabla \times (\nabla \times \underline{u})$

$$= \frac{\partial}{\partial y}\left[\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right] - \frac{\partial}{\partial z}\left[\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right]$$

$\underline{x}$ component of $\nabla(\nabla . \underline{u})$

$$= \frac{\partial}{\partial x}\left(\cancel{\frac{\partial u_x}{\partial x}} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)$$

$\underline{x}$ component of $-\nabla^2 \underline{u}$

$$= -\left(\cancel{\frac{\partial^2 u_x}{\partial x^2}} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)$$

Since $\frac{\partial^2 u_y}{\partial x \partial y} = \frac{\partial^2 u_y}{\partial y \partial x}$, etc, this

demonstrates that for the $\hat{\underline{x}}$ components

$$\nabla \times (\nabla \times \underline{u}) = \nabla(\nabla . \underline{u}) - \nabla^2 \underline{u}$$

Q2 (a) $\nabla \cdot \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

(in 3-dimensions)

(b) $r^n = (x^2 + y^2 + z^2)^{n/2}$

$\hat{\underline{x}}$ component of $\nabla(r^n)$

$$= \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{n/2}$$

$$= \frac{n}{2}\left(x^2+y^2+z^2\right)^{\left(\frac{n-2}{2}\right)} 2x = n r^{n-2} x$$

and similarly for the $\hat{\underline{y}}$ and $\hat{\underline{z}}$ components

$$\therefore \quad \nabla(r^n) = n r^{n-2} \underline{r}$$

For example, $n=-1$: $\nabla\left(\frac{1}{r}\right) = -\frac{\underline{r}}{r^3}$

Q2 (c) Using answer to Q1(a) with $\phi = r^n$

and $\nabla \times \underline{a} = 0$ since $\underline{a}$ is constant.

$$\nabla \times (r^n \underline{a}) = -\underline{a} \times [\nabla(r^n)]$$

using Q2 (b)

$$= -\underline{a} \times [n r^{n-2} \underline{r}]$$

$$= n r^{n-2} \underline{r} \times \underline{a}$$

(d) $(\underline{a} . \nabla) \underline{r}$

$$= \left[a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}\right]\left[\hat{\underline{x}} x + \hat{\underline{y}} y + \hat{\underline{z}} z\right]$$

$$= \hat{\underline{x}} a_x + \hat{\underline{y}} a_y + \hat{\underline{z}} a_z$$

$$= \underline{a}$$

Q2(e) Using answer to Q1(b) with $\underline{u} = \underline{a}$, and $\underline{v} = \underline{r}$

$$(\underline{r}.\nabla)\underline{a} = \underline{r}(\nabla.\underline{a}) = 0$$

since $\underline{a}$ is a constant vector.

$$-(\underline{a}.\nabla)\underline{r} = -\underline{a} \qquad \text{from Q2(d)}$$

$$\underline{a}(\nabla.\underline{r}) = 3\underline{a} \qquad \text{from Q2(a)}$$

$$\therefore \quad \nabla \times (\underline{a} \times \underline{r}) = -\underline{a} + 3\underline{a} = 2\underline{a}$$

(f) Using Vector Identity (3) on the Formula Sheet

$$\nabla(r^n \underline{a}.\underline{r}) = \nabla(r^n)\,\underline{a}.\underline{r} + r^n \nabla(\underline{a}.\underline{r})$$

Using Vector Identity (4) and

$$\nabla \times \underline{a} = \nabla \times \underline{r} = (\underline{r}.\nabla)\underline{a} = 0$$

$$(\underline{a}.\nabla)\underline{r} = \underline{a}$$

$$\therefore \nabla(r^n \underline{a}.\underline{r}) = \underbrace{n r^{n-2}\underline{r}}_{\text{using Q2(b)}}(\underline{a}.\underline{r}) + r^n \underline{a}$$

$$= r^{n-2}\left(n(\underline{a}.\underline{r})\underline{r} + r^2 \underline{a}\right)$$

Q2(g) Using answer to Q1(b) with

$\underline{u} = r^n \underline{a}$ and $\underline{v} = \underline{r}$

(i) $(\underline{r} \cdot \nabla) r^n \underline{a} = \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right] \left[x^2+y^2+z^2 \right]^{n/2} \underline{a}$

$= \frac{n}{2} \cdot 2 \cdot \left[x^2+y^2+z^2 \right] \left[x^2+y^2+z^2 \right]^{\frac{n-2}{2}} \underline{a}$

$= n r^n \underline{a}$

(ii) $-(r^n \underline{a} \cdot \nabla) \underline{r} = -r^n \underline{a}$ using Q2(d)

(iii) $r^n \underline{a} (\nabla \cdot \underline{r}) = 3 r^n \underline{a}$ using Q2(a)

Using Vector Identity (5) from Useful Formula Sheet

and $\nabla \cdot \underline{a} = 0$

(iv) $-\underline{r}(\nabla \cdot [r^n \underline{a}]) = -\underline{r}(\underline{a} \cdot \nabla (r^n))$

$= -\underline{r}(n r^{n-2} (\underline{a} \cdot \underline{r}))$ using Q2(b)

Total (i) + (ii) + (iii) + (iv)

$$\nabla \times (r^n \underline{a} \times \underline{r}) = (n+2) r^n \underline{a} - n r^{n-2} (\underline{a} \cdot \underline{r}) \underline{r}$$

$$= r^{n-2} \left((n+2) r^2 \underline{a} - n (\underline{a} \cdot \underline{r}) \underline{r} \right)$$