PHYS30441

Electrodynamics

Exercises: Revision of vector calculus

Solutions - Terry Wyatt.

(b) Using general index notation for vectors

A few preliminaries on index notation for vectors in cartesian coordinates

We shall write the "i'th" component of a vector $\underline{u}$ in 3-dimensions as:

$$u_i = [\underline{u}]_i \quad , \quad \text{where } i = 1, 2, 3$$

and, in particular, $x_i = [\underline{r}]_i$

Because the three coordinates are orthogonal we have

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \begin{cases} = 1 & \text{if } i = j \\ = 0 & \text{if } i \neq j \end{cases}$$

If an index is repeated in an expression it is summed over (unless explicitly stated to the contrary) so that we can write:

e.g. $$\nabla . \underline{u} = \frac{\partial u_j}{\partial x_j}$$

$$\underline{u} \times \underline{v} = \varepsilon_{ijk} \hat{\underline{x}}_i u_j v_k$$

where

$$1 = \varepsilon_{123} = \underbrace{\varepsilon_{312} = \varepsilon_{231}}_{\text{cyclic permutations}} = -\varepsilon_{321} \text{ and other anti-cyclic permutations.}$$

$\therefore \varepsilon_{ijk} = 0$ if any pair of indices are equal.

In the proofs set in the exercises the following relation will be useful

$$\varepsilon_{ijk}\,\varepsilon_{kmn} = \varepsilon_{ijk}\,\varepsilon_{mnk} = \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)$$

Q1 (a) Considering the i'th component:

$$[\nabla \times (\phi \underline{u})]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\phi u_k)$$

$$= \varepsilon_{ijk} \left[\frac{\partial \phi}{\partial x_j} u_k + \phi \frac{\partial u_k}{\partial x_j} \right]$$

$$= \left[(\nabla \phi) \times \underline{u} + \phi (\nabla \times \underline{u}) \right]_i$$

$$= \left[\phi (\nabla \times \underline{u}) - \underline{u} \times (\nabla \phi) \right]_i$$

Q1 (b)

$$[\nabla \times (\underline{u} \times \underline{v})]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} [\underline{u} \times \underline{v}]_k$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\varepsilon_{kmn} u_m v_n \right)$$

$$= \left[\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \right] \frac{\partial}{\partial x_j} \left(u_m v_n \right)$$

$$= \frac{\partial}{\partial x_j} \left(u_i v_j - u_j v_i \right)$$

$$= v_j \frac{\partial u_i}{\partial x_j} + \frac{\partial v_j}{\partial x_j} u_i - \frac{\partial u_j}{\partial x_j} v_i - u_j \frac{\partial v_i}{\partial x_j}$$

$$= \left[(\underline{v} \cdot \nabla) \underline{u} + (\nabla \cdot \underline{v}) \underline{u} - (\nabla \cdot \underline{u}) \underline{v} - (\underline{u} \cdot \nabla) \underline{v} \right]_i$$

Q1(c)

(i) $$\left[\frac{1}{2}\nabla(\underline{u}\cdot\underline{u})\right]_i = \frac{1}{2}\frac{\partial}{\partial x_i}\left(u_j u_j\right)$$

$$= \frac{2}{2} u_j \frac{\partial u_j}{\partial x_i} = u_j \frac{\partial u_j}{\partial x_i}$$

(ii) $$\left[(\underline{u}\cdot\nabla)\underline{u}\right]_i = u_j \frac{\partial u_i}{\partial x_j}$$

(iii) $$\left[\underline{u}\times(\nabla\times\underline{u})\right]_i = \varepsilon_{ijk} u_j \left[\nabla\times\underline{u}\right]_k$$

$$= \varepsilon_{ijk} u_j \varepsilon_{kmn} \frac{\partial u_n}{\partial x_m}$$

$$= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) u_j \frac{\partial u_n}{\partial x_m}$$

$$= u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j}$$

(ii) + (iii) $= u_j \dfrac{\partial u_j}{\partial x_i} =$ (i) , as required

Q1(d)

$$\left[\nabla \times (\nabla \times \underline{u})\right]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left[\nabla \times \underline{u}\right]_k$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{kmn} \frac{\partial u_n}{\partial x_m}$$

$$= \left[\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right] \frac{\partial}{\partial x_j}\left(\frac{\partial u_n}{\partial x_m}\right)$$

$$= \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial^2 u_i}{\partial x_j^2}$$

$$= \left[\nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}\right]_i$$

Q2 (a) $\underline{\nabla} \cdot \underline{r} = \dfrac{\partial x_j}{\partial x_j} = 3$

(b) $r^n = (x_j x_j)^{n/2}$

$$\left[\nabla(r^n)\right]_i = \frac{\partial}{\partial x_i}(x_j x_j)^{n/2} = \frac{n}{2}(x_k x_k)^{\frac{n-2}{2}} \delta_{ij} 2x_j$$

$$= n r^{n-2} x_i = \left[n r^{n-2} \underline{r}\right]_i$$

(c) $\left[\nabla \times (r^n \underline{a})\right]_i = \varepsilon_{ijk} \dfrac{\partial}{\partial x_j}(r^n a_k)$

$$= \varepsilon_{ijk} \left\{ \underbrace{\left(\frac{\partial [r^n]}{\partial x_j}\right)}_{= n r^{n-2} x_j} a_k + r^n \underbrace{\frac{\partial a_k}{\partial x_j}}_{= 0} \right\}$$

from answer to Q2(b)

$$= n r^{n-2} \varepsilon_{ijk} x_j a_k = n r^{n-2} \left[\underline{r} \times \underline{a}\right]_i$$

Q2 (d) $\left[(\underline{a}.\nabla)\underline{r}\right]_i = a_j \dfrac{\partial x_i}{\partial x_j}$

$= a_j \delta_{ij} = a_i = \left[\underline{a}\right]_i$

(e) $\left[\nabla \times (\underline{a} \times \underline{r})\right]_i = \varepsilon_{ijk} \dfrac{\partial}{\partial x_j}\left[\underline{a} \times \underline{r}\right]_k$

$= \varepsilon_{ijk} \dfrac{\partial}{\partial x_j}\left(\varepsilon_{kmn}\, a_m x_n\right)$

$= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) \dfrac{\partial}{\partial x_j}\left(a_m x_n\right)$

$= \dfrac{\partial}{\partial x_j}\left(a_i x_j\right) - \dfrac{\partial}{\partial x_j}\left(a_j x_i\right)$

$= a_i \dfrac{\partial x_j}{\partial x_j} - a_j \delta_{ij}$ since $\underline{a}$ is a constant and $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$

$= 3a_i - a_i$

$= 2a_i = 2\left[\underline{a}\right]_i$