Lecture 11) Special Relativity in the Minkowski Representation (index notation)

Motivated by the relative minus sign in the scalar product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

we define

$$a^\mu \equiv (a^0, a^1, a^2, a^3) \quad \text{a } \underline{\text{contravariant}} \text{ vector} \quad (11.1)$$

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (a^0, -a^1, -a^2, -a^3) \quad (11.2)$$

a <u>covariant</u> vector

Example $\quad x^\mu = (ct, x, y, z) = (ct, \underline{r})$

$$x_\mu = (ct, -x, -y, -z) = (ct, -\underline{r})$$

The scalar product is written

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a^{\mu} b_{\mu} = a_{\mu} b^{\mu}$$

Notes 1) Repeated indices are summed over
("Einstein summation convention")

2) Only if there is one upper and one lower index is the result Lorentz invariant.

We can convert contravariant $\Longleftrightarrow$ covariant vectors using metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

$$\left.\begin{array}{lll} g^{\mu\nu} a_\nu = a^\mu & \text{raising Lorentz index} \\ g_{\mu\nu} a^\nu = a_\mu & \text{lowering} \quad " \quad\quad " \end{array}\right\} \text{verify these explicitly.}$$

Exercises

Verify that

1) $$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\nu = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times4}$$

2) $x^\mu x^\mu$ is <u>not</u> a Lorentz invariant

Lecture 10) Special Relativity and 4-vectors

(reminder)

Postulates The laws of physics (results of experiments) are the same in all inertial frames of reference

including measuring the speed of light in vacuum.

Lorentz Transformation

Transform from coordinates in inertial frame, S, to another, S', moving with respect to S with velocity $\underline{v} = \beta c \hat{\underline{x}}$

$$\boxed{\begin{aligned} [ct'] &= \gamma\left([ct] - \beta x\right) \\ x' &= \gamma\left(x - \beta[ct]\right) \\ y' &= y \\ z' &= z \end{aligned}}$$

Minkowski representation of Lorentz Transformations

Consider the contravariant 4-vector $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

and $\underline{\beta} = \beta \hat{\underline{x}}$

$$\left.\begin{aligned}
x'^0 &= \gamma(x^0 - \beta x^1) = \underbrace{\frac{\partial x'^0}{\partial x^0}}_{\gamma} + \underbrace{\frac{\partial x'^0}{\partial x^1}}_{-\gamma\beta} + \underbrace{\frac{\partial x'^0}{\partial x^2}}_{0} + \cdots \\
x'^1 &= \gamma(-\beta x^0 + x^1) = \underbrace{\frac{\partial x'^1}{\partial x^0}}_{-\gamma\beta} + \cdots \\
x'^2 &= x^2 \\
x'^3 &= x^3
\end{aligned}\right\} (11.5)$$

Writing in more compact form

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = \Lambda^\mu{}_\nu x^\nu = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad (11.6)$$

Verify (11.6) equivalent to (11.5).

How does a covariant 4-vector transform?

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$$

$$x_0' = ct' = \gamma(ct - \beta x) = \gamma(x_0 + \beta x_1) = \underbrace{\frac{\partial x_0'}{\partial x_0}}_{\gamma} x_0 + \underbrace{\frac{\partial x_0'}{\partial x_1}}_{\gamma_\beta} x_1 + \ldots$$

$$x_1' = -x' = -\gamma(-\beta ct + x) = \gamma(\beta x_0 + x_1) = \ldots \qquad (11.7)$$

or $x_\mu' = \dfrac{\partial x_\mu'}{\partial x_\nu} x_\nu = \left(\Lambda^{-1}\right)^\nu{}_\mu x_\nu$

where $\left(\Lambda^{-1}\right)^\nu{}_\mu = \left(\begin{array}{cc|cc} \gamma & \gamma_\beta & 0 & 0 \\ \gamma_\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad (11.8)$

Verify this!

Exercises

Verify the following:

1) $$\Lambda^{\mu}{}_{\nu}\left(\Lambda^{-1}\right)^{\nu}{}_{\lambda} = \delta^{\mu}{}_{\lambda} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times4}$$

$S' \to S$ transformations

2) $$x^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} x'^{\nu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\nu} x'^{\nu} \qquad (11.9)$$

3) $$x_{\mu} = \frac{\partial x_{\mu}}{\partial x'_{\nu}} x'_{\nu} = \Lambda^{\nu}{}_{\mu} x'_{\nu} \qquad (11.10)$$

Let's define a 4-dimensional "grad" operator and work out how this transforms

Reminder

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\mu} \, !$$

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} = \underset{\substack{\uparrow \\ \text{from } (11.9)}}{\left(\Lambda^{-1}\right)^{\nu}{}_{\mu}} \frac{\partial}{\partial x^{\nu}}$$

$$\frac{\partial}{\partial x'_{\mu}} = \frac{\partial x_{\nu}}{\partial x'_{\mu}} \frac{\partial}{\partial x_{\nu}} = \underset{\substack{\uparrow \\ \text{from } (11.10)}}{\Lambda^{\mu}{}_{\nu}} \frac{\partial}{\partial x_{\nu}}$$

That is: differential w.r.t. $\left\{\begin{matrix}\text{contravariant } x^{\mu} \\ \text{covariant } x_{\nu}\end{matrix}\right\}$ transforms as $\left\{\begin{matrix}\text{covariant} \\ \text{contravariant}\end{matrix}\right\}$!

Let's define the following shorthand:

$$\boxed{\begin{aligned}\partial_\mu &= \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)\\ \partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)\end{aligned}}$$