$$\left.\begin{array}{lll} g^{\mu\nu} a_\nu = a^\mu & \text{raising Lorentz index} \\ g_{\mu\nu} a^\nu = a_\mu & \text{lowering} \quad " \quad\quad " \end{array}\right\} \text{verify these explicitly.}$$

<u>Exercises</u>

Verify that

1) $$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\nu = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times 4}$$

2) $x^\mu x^\mu$ is <u>not</u> a Lorentz invariant

Minkowski representation of Lorentz Transformations

Consider the contravariant 4-vector $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

and $\underline{\beta} = \beta \hat{\underline{x}}$

$$\left.\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) = \underbrace{\frac{\partial x'^0}{\partial x^0}}_{\gamma} + \underbrace{\frac{\partial x'^0}{\partial x^1}}_{-\gamma\beta} + \underbrace{\frac{\partial x'^0}{\partial x^2}}_{0} + \cdots \\ x'^1 &= \gamma(-\beta x^0 + x^1) = \underbrace{\frac{\partial x'^1}{\partial x^0}}_{-\gamma\beta} + \cdots \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}\right\} (11.5)$$

Writing in more compact form

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = \Lambda^\mu{}_\nu x^\nu = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & \\ -\gamma\beta & \gamma & & \\ \hline & & 1 & 0 \\ 0 & & 0 & 1 \end{array}\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad (11.6)$$

Verify (11.6) equivalent to (11.5).

How does a covariant 4-vector transform?

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$$

$$x_0' = ct' = \gamma(ct - \beta x) = \gamma(x_0 + \beta x_1) = \underbrace{\frac{\partial x_0'}{\partial x_0}}_{\gamma} x_0 + \underbrace{\frac{\partial x_0'}{\partial x_1}}_{\gamma\beta} x_1 + \ldots$$

$$x_1' = -x' = -\gamma(-\beta ct + x) = \gamma(\beta x_0 + x_1) = \ldots \qquad (11.7)$$

or $$x_\mu' = \frac{\partial x_\mu'}{\partial x_\nu} x_\nu = (\Lambda^{-1})^\nu{}_\mu x_\nu$$

where $$(\Lambda^{-1})^\nu{}_\mu = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & \multicolumn{2}{c}{0} \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right) \qquad (11.8)$$

Verify this!

Exercises

Verify the following:

1) $$\Lambda^{\mu}{}_{\nu}\left(\Lambda^{-1}\right)^{\nu}{}_{\lambda} = \delta^{\mu}{}_{\lambda} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times4}$$

$S' \to S$ transformations

2) $$x^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} x'^{\nu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\nu} x'^{\nu} \qquad (11.9)$$

3) $$x_{\mu} = \frac{\partial x_{\mu}}{\partial x'_{\nu}} x'_{\nu} = \Lambda^{\nu}{}_{\mu} x'_{\nu} \qquad (11.10)$$