$$\left.\begin{array}{lll} g^{\mu\nu} a_\nu = a^\mu & & \text{raising Lorentz index} \\ g_{\mu\nu} a^\nu = a_\mu & & \text{lowering} \quad " \quad\quad " \end{array}\right\} \text{verify these explicitly.}$$

Exercises

Verify that

1) $$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\nu = \begin{bmatrix} 1 & & 0 & \\ & 1 & & \\ 0 & & 1 & \\ & & & 1 \end{bmatrix} = \mathbb{1}_{4\times 4}$$

2) $x^\mu x^\mu$ is <u>not</u> a Lorentz invariant

Writing out the Lecture 11 exercises as matrices

$$g^{\mu\nu} a_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = a^\mu$$

$$g_{\mu\nu} a^\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} a^0 \\ -a^1 \\ -a^2 \\ -a^3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_\mu$$

$$g^{\mu\nu} g_{\nu\lambda} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$a^\mu b^\mu = a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$= \gamma\left(a'^0 + \beta a'^1\right)\gamma\left(b'^0 + \beta b'^1\right) + \gamma\left(a'^1 + \beta a'^0\right)\gamma\left(b'^1 + \beta b'^0\right) + a'^2 b'^2 + a'^3 b'^3$$

$$= \gamma^2\left[\left(1+\beta^2\right)\left(a'^0 b'^0 + a'^1 b'^1\right) + 2\beta\left(a'^0 b'^1 + a'^1 b'^0\right) + a'^2 b'^2 + a'^3 b'^3\right.$$

$$\neq a'^\mu b'^\mu$$

$\therefore$ $a^\mu b^\mu$ is not a Lorentz scalar

Minkowski representation of Lorentz Transformations

Consider the contravariant 4-vector $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

and $\underline{\beta} = \beta \hat{\underline{x}}$

$$\left.\begin{aligned}
x'^0 &= \gamma(x^0 - \beta x^1) = \underbrace{\frac{\partial x'^0}{\partial x^0}}_{\gamma} + \underbrace{\frac{\partial x'^0}{\partial x^1}}_{-\gamma\beta} + \underbrace{\frac{\partial x'^0}{\partial x^2}}_{0} + \cdots \\
x'^1 &= \gamma(-\beta x^0 + x^1) = \underbrace{\frac{\partial x'^1}{\partial x^0}}_{-\gamma\beta} + \cdots \\
x'^2 &= x^2 \\
x'^3 &= x^3
\end{aligned}\right\} (11.5)$$

Writing in more compact form

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = \Lambda^\mu{}_\nu x^\nu = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad (11.6)$$

Verify (11.6) equivalent to (11.5).

$$x'^{\mu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \gamma x^0 - \gamma\beta x^1 \\ -\gamma\beta x^0 + \gamma x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

How does a covariant 4-vector transform?

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$$

$$x_0' = ct' = \gamma(ct - \beta x) = \gamma(x_0 + \beta x_1) = \underbrace{\frac{\partial x_0'}{\partial x_0}}_{\gamma} x_0 + \underbrace{\frac{\partial x_0'}{\partial x_1}}_{\gamma\beta} x_1 + \ldots$$

$$x_1' = -x' = -\gamma(-\beta ct + x) = \gamma(\beta x_0 + x_1) = \ldots \qquad (11.7)$$

or $$x_\mu' = \frac{\partial x_\mu'}{\partial x_\nu} x_\nu = \left(\Lambda^{-1}\right)^\nu{}_\mu x_\nu$$

where $$\left(\Lambda^{-1}\right)^\nu{}_\mu = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) \qquad (11.8)$$

Verify this !

Writing out each <u>non-zero</u> term using index notation

$$x'_0 = (\Lambda^{-1})^0{}_0 x_0 + (\Lambda^{-1})^1{}_0 x_1 = \gamma x_0 + \gamma\beta x_1$$

$$x'_1 = (\Lambda^{-1})^0{}_1 x_0 + (\Lambda^{-1})^1{}_1 x_1 = \gamma\beta x_0 + \gamma x_1$$

$$x'_2 = (\Lambda^{-1})^2{}_2 x_2 = x_2$$

$$x'_3 = (\Lambda^{-1})^3{}_3 x_3 = x_3$$

Exercises

Verify the following:

1) $$\Lambda^{\mu}{}_{\nu}\left(\Lambda^{-1}\right)^{\nu}{}_{\lambda} = \delta^{\mu}{}_{\lambda} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times 4}$$

$S' \to S$ transformations

2) $$x^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} x'^{\nu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\nu} x'^{\nu} \qquad (11.9)$$

3) $$x_{\mu} = \frac{\partial x_{\mu}}{\partial x'_{\nu}} x'_{\nu} = \Lambda^{\nu}{}_{\mu} x'_{\nu} \qquad (11.10)$$

$$\Lambda^{\mu}{}_{\nu}\left(\Lambda^{-1}\right)^{\nu}{}_{\lambda} = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} \gamma^2(1-\beta^2) & \gamma^2(\beta-\beta) & & \\ \gamma^2(-\beta+\beta) & \gamma^2(-\beta^2+1) & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, x'^{\nu} = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma x'^0 + \gamma\beta x'^1 \\ \gamma\beta x'^0 + \gamma x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$= x^{\mu}, \text{ as required by Eqn. (11.9)}$$

<u>Using index notation</u>: $x_\mu = \Lambda^{\nu}{}_{\mu}\, x'_\nu$ may be written as

$$x_0 = \Lambda^{\nu}{}_{0}\, x'_\nu = \Lambda^{0}{}_{0}\, x'_0 + \Lambda^{1}{}_{0}\, x'_1 \quad \left(\text{since } \Lambda^{2}{}_{0} = \Lambda^{3}{}_{0} = 0\right)$$

$$= \gamma x'_0 - \gamma\beta x'_1$$

$$x_1 = \Lambda^{\nu}{}_{1}\, x'_\nu = \Lambda^{0}{}_{1}\, x'_0 + \Lambda^{1}{}_{1}\, x'_1 = -\gamma\beta x'_0 + \gamma x'_1$$

$$x_2 = \Lambda^{\nu}{}_{2}\, x'_\nu = \Lambda^{2}{}_{2}\, x'_2 = x'_2, \text{ and similarly } x_3 = x'_3.$$

<u>Alternatively</u>, writing x'_ν as a row vector* for matrix multiplication

$$\begin{pmatrix} x'_0 & x'_1 & x'_2 & x'_3 \end{pmatrix} \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & & 0 \\ -\gamma\beta & \gamma & & \\ \hline & & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} \gamma x'_0 - \gamma\beta x'_1, & -\gamma\beta x'_0 + \gamma x'_1 & x'_2 & x'_3 \end{pmatrix}$$

Both methods give the correct transformation equations for x_μ in terms of x'_μ. (cf equations 11.7 for x'_μ in terms of x_μ)

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* This would be the "natural" way, given that in $\Lambda^{\nu}{}_{\mu}$ index ν labels <u>rows</u>!