Lecture 12 (b) 4-vectors for electrodynamics

1) 4-momentum of a particle

Define $p^\mu = m u^\mu = m \dfrac{dx^\mu}{d\tau} = \gamma m (c, \underline{v})$

$\uparrow$

mass of particle in rest frame

"rest mass", "invariant mass", "mass".

p^μ <u>must</u> be a 4-vector since u^μ is a 4-vector and m is by definition a Lorentz-invariant

Components of p^μ

For $i = 1, 2, 3$: $p^i = \gamma m v^i$

The components of the relativistic momentum.

$$p^0 = \gamma m c = \frac{\gamma m c^2}{c} = \frac{E}{c}$$

where $E = \gamma m c^2$ is the total relativistic energy

$$\Longrightarrow \quad p^\mu = \left(\frac{E}{c}, \underline{p} \right)$$

Lorentz Transformation

$$p'^\mu = \Lambda^\mu{}_\nu \, p^\nu$$

Magnitude

In rest frame $\gamma = 1$, $\underline{v} = 0$

$$p^\mu = m(c, 0)$$

$$p^\mu p_\mu = (mc)^2$$

This must be true in all frames of reference

$$\left(\frac{E}{c}\right)^2 - p^2 = (mc)^2$$

$$E^2 - (pc)^2 = (mc^2)^2$$

2) 4 - current

Define $j^\mu = \rho_0 u^\mu$

↑ the charge density in the rest frame of the charge distribution

$\therefore$ j^μ <u>must</u> be a 4-vector

u^μ is a 4-vector

ρ_0 is by definition invariant.

Components of j^μ

For $i = 1, 2, 3$ $\quad j^i = \underbrace{\gamma\rho_0}_{} v^i$

is the charge density in frame in which charge distribution is moving. (ρ)

Exercise: prove this!

$$j^0 = \gamma\rho_0 c$$

$$\Rightarrow \quad j^\mu = (\gamma\rho_0 c, \gamma\rho_0 \underline{v}) = (\rho c, \rho\underline{v}) = (\rho c, \underline{j})$$

An application of this new notation

The continuity equation

$$\frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \nabla \cdot \underline{j} = 0$$

can be written as

$$\frac{\partial j^\mu}{\partial x^\mu} = \boxed{\partial_\mu j^\mu = 0}$$

Product of upper and lower index guarantees that

$$\partial'_\mu j'^\mu = 0$$

<u>Lorentz covariance</u> Laws of physics take the same form as required by Special Relativity.

The Wave Equation in Lorentz-covariant form

We can write the D'Alembertian

$$\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box^2$$

Product of upper and lower indices guarantees that $\Box^2$ is a Lorentz invariant

Wave equations

$$\left.\begin{aligned} \Box^2\left(\frac{V}{c}\right) &= \frac{\rho}{c\,\varepsilon_0} = \mu_0(\rho c) \\ \Box^2 \underline{A} &= \mu_0 \underline{j} \end{aligned}\right\} \text{R.H.S. is a constant} \times \text{4-vector } j^\mu$$

L.H.S. is Lorentz invariant $\times$ something that <u>must</u> be a 4-vector!

Let's define : $A^\mu = \left(\frac{V}{c}, \underline{A} \right)$

Then we can write

$$\Box^2 A^\mu = \mu_0 j^\mu$$

guarantees that A^μ is a 4-vector !

The retarded potentials (9.8) & (9.9)

$$\frac{V}{c} = \frac{1}{4\pi\varepsilon_0 c}\int_{V} \frac{\cancel{c}\,\rho(\mathbf{r}', t_{ret})}{R}\, d\tau'$$

$$A^0 = \frac{\mu_0}{4\pi}\int_{V} \frac{j^0(\mathbf{r}', t_{ret})}{R}\, d\tau'$$

$$\boxed{A^\mu = \frac{\mu_0}{4\pi}\int_{V} \frac{j^\mu(\mathbf{r}', t_{ret})}{R}\, d\tau'}$$