Lecture 13) More Lorentz - covariant notatation for Electrodynamics

The Lorentz ~~Gauge Condition~~

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{V}{c}\right) + \nabla \cdot \underline{A} = 0$$

$$\boxed{\partial_\mu A^\mu = 0}$$

is manifestly Lorentz - covariant.

General Gauge Transformation

$$\frac{V}{c} \Longrightarrow \frac{V}{c} - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

$$\underline{A} \Longrightarrow \underline{A} + \nabla \psi$$

Look at the components $(i = 1, 2, 3)$

$$A^i \Rightarrow A^i + \frac{\partial \psi}{\partial x^i} = A^i - \frac{\partial \psi}{\partial x_i} = A^i - \partial^i \psi$$

NB Take careful note of signs and upper - lower indices.

$$\boxed{A^\mu \Rightarrow A^\mu - \partial^\mu \psi}$$

What about $\underline{E}$ & $\underline{B}$ fields?

$$\frac{\underline{E}}{c} = -\nabla\left(\frac{V}{c}\right) - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}$$

$(i = 1, 2, 3)$

$$\frac{E_i}{c} = -\frac{\partial A^0}{\partial x^i} - \frac{\partial}{\partial x^0} A^i = \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0}$$

$$= \partial^i A^0 - \partial^0 A^i$$

$$= F^{i0} = -F^{0i}$$

$$B_i = [\nabla \times \underline{A}]_i \qquad i,j,k = 1,2,3$$

$$= \frac{\partial}{\partial x^j} A^k - \frac{\partial}{\partial x^k} A^j = -\frac{\partial A^k}{\partial x_j} + \frac{\partial A^j}{\partial x_k} = -\partial^j A^k + \partial^k A^j$$

$$= -F^{jk} = F^{kj}$$

e.g. $B_1 = -\partial^2 A^3 + \partial^3 A^2 = +F^{32} = -F^{23}$

Putting this together

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{array}{c} \mu\downarrow \ \nu\rightarrow \\ \begin{bmatrix} 0 & -\frac{E_1}{c} & -\frac{E_2}{c} & -\frac{E_3}{c} \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ \frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix} \end{array}$$

The Electromagnetic Field Tensor

There are only six independent terms: $\underline{E}$, $\underline{B}$

Exercise: Verify explicitly each element of $F^{\mu\nu}$!

Lorentz Tranformations of $F^{\mu\nu}$

Since $F^{\mu\nu}$ is a Lorentz tensor of rank 2

(It has 2 contravariant indicies)

so we need 2 Λ matrices!

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

Looks horrendous but index notation makes these calculations relatively simple!

Reminder

$$\Lambda^{\delta}{}_{\gamma} = \begin{array}{c} \\ \delta \downarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \gamma \longrightarrow \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

$$F^{\alpha\beta} = \begin{array}{c} \\ \alpha \downarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \beta \longrightarrow \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \end{array}$$

Look at specific examples

$$\frac{E_1'}{c} = F'^{10} = \Lambda'^{1}{}_{\alpha} \Lambda'^{0}{}_{\beta} F^{\alpha\beta}$$

We can ignore terms with $\alpha = \beta$ $\because$ $F^{\alpha\alpha} = 0$

$$\frac{E_1'}{c} = \Lambda^{1}{}_{0} \Lambda^{0}{}_{1} F^{01} + \Lambda^{1}{}_{1} \Lambda^{0}{}_{0} F^{10}$$

are the only two non-zero terms!

$$= (-\gamma\beta)^2 \left(-\frac{E_1}{c}\right) + \gamma^2 \left(\frac{E_1}{c}\right)$$

$$= \gamma^2 (1 - \beta^2) \frac{E_1}{c} = \frac{E_1}{c}$$

$$\frac{E_2'}{c} = F'^{20} = \Lambda^2{}_\alpha \Lambda^0{}_\beta F^{\alpha\beta}$$

$$= \Lambda^2{}_2 \left(\Lambda^0{}_0 F^{20} + \Lambda^0{}_1 F^{21} \right)$$

$$= \gamma \left(\frac{E_2}{c} - \beta B_3 \right)$$

Exercise: work out explicitly the other transformations.