Lecture 15) $\underline{E}$ and $\underline{B}$ fields for a moving point charge q.

In rest frame S'

$$\frac{\underline{\underline{E}}}{c} = \frac{q}{4\pi\varepsilon_0 c}\frac{\underline{R}'}{(R')^3}$$

$$\frac{E_1'}{c} = \frac{q}{4\pi\varepsilon_0 c}\frac{x'^1}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\frac{\gamma\left[x^1 - \beta x^0\right]}{(R')^3} \qquad (15.1)$$

$$\left.\begin{aligned} \frac{E_2'}{c} &= \qquad = \frac{q}{4\pi\varepsilon_0 c}\frac{x^2}{(R')^3} \\ \frac{E_3'}{c} & \qquad\quad = \quad ? \end{aligned}\right\} \qquad (15.2)$$

Exercise : Fill in the blanks!

<u>Method 1</u> From expression for A^μ in frame S (Lecture 14,

and rules for finding $F^{\mu\nu}$ from A^μ (Lecture 1

$$\frac{E_1}{c} = F^{10} = \partial^1 A^0 - \partial^0 A^1 = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0}$$

$$= -\frac{q\gamma}{4\pi\varepsilon_0 c}\left[\frac{\left(-\frac{1}{2}\right)2\gamma^2[x^1-\beta x^0]}{(R')^3} + \frac{\beta\left(-\frac{1}{2}\right)2\gamma^2[x^1-\beta x^0](-\beta)}{(R')^3}\right]$$

$$= \frac{q\gamma}{4\pi\varepsilon_0 c}\frac{[x^1-\beta x^0]}{(R')^3}\underbrace{\gamma^2(1-\beta^2)}_{=1} \qquad (15.3)$$

$$\frac{E_2}{c} = F^{20} = \partial^2 A^0 - \partial^0 A^2 = -\frac{\partial A^0}{\partial x^2} = \qquad = \frac{q\gamma}{4\pi\varepsilon_0 c}\frac{x^2}{(R')^3}$$

$$\frac{E_3}{c} = \qquad = \qquad = \qquad = \qquad = \qquad \frac{x^3}{}$$

Fill in the blanks!

(15.4

Potentials in frame S (From Lecture 14) Reminder

L.T. $\frac{V}{c} = A^0 = \gamma(A'^0 + \beta A'^1) = \gamma A'^0$

(Note: inverse Lorentz Transformation.)

$$\therefore \quad A^0 = \frac{q}{4\pi\varepsilon_0 c}\gamma \frac{1}{\left[\left(\gamma[x^1-\beta x^0]\right)^2+(x^2)^2+(x^3)^2\right]^{1/2}}$$

$$A^1 = \gamma(\beta A'^0 + A'^1) = \gamma\beta A'^0 = \beta A^0$$

$$= \frac{q}{4\pi\varepsilon_0 c}\gamma\beta \frac{1}{\left[\left(\gamma[x^1-\beta x^0]\right)^2+(x^2)^2+(x^3)^2\right]^{1/2}}$$

$$A^2 = A^3 = 0$$

Exercise for you: Cross-check that $A'^\mu A'_\mu = A^\mu A_\mu$.

Method 2

Use inverse transformations from Lecture 13 $\left(S' \Leftrightarrow S,\ \beta \Rightarrow -\beta\right)$ on (15.1) & (15.2) with $\underline{\beta}' = 0$

$$\left.\begin{aligned} E_1 &= E_1' \\ E_2 &= \gamma E_2' \\ E_3 &= \gamma E_3' \end{aligned}\right\} \quad (15.5)$$

(15.5) consistent with (15.3) & (15.4).

Lecture 15) E and B Field Transformations

Lorentz transformation is more complicated for tensors of rank 2:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

For each specific element of $F'^{\mu\nu}$, the sums over α, β yield only one or two non-zero terms! Alteratively, writing in matrix form:

$$F' = \Lambda F \Lambda^{T}$$

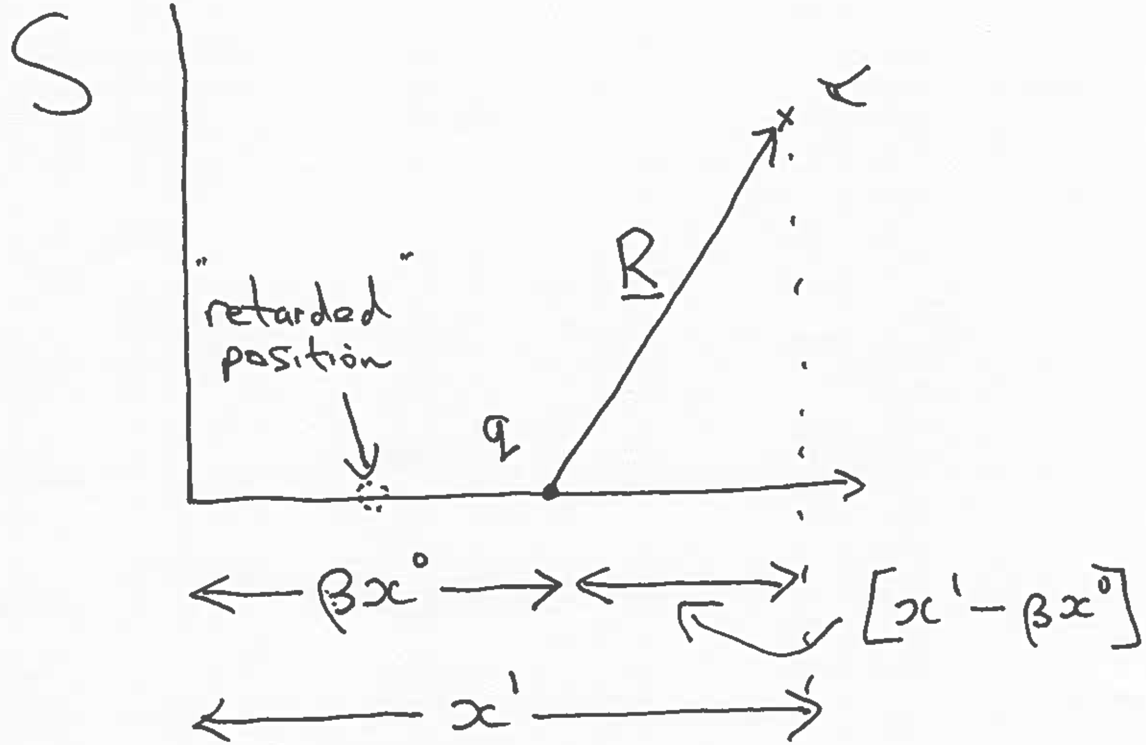
it can be shown that:

$$\boxed{\begin{aligned} \frac{E_1'}{c} &= \frac{E_1}{c} & \qquad B_1' &= B_1 \\ \frac{E_2'}{c} &= \gamma\left(\frac{E_2}{c} - \beta B_3\right) & \qquad B_2' &= \gamma\left(B_2 + \beta \frac{E_3}{c}\right) \\ \frac{E_3'}{c} &= \gamma\left(\frac{E_3}{c} + \beta B_2\right) & \qquad B_3' &= \gamma\left(B_3 - \beta \frac{E_2}{c}\right) \end{aligned}}$$

Applying inverse transformations $S' \Rightarrow S$, $\beta \Rightarrow -\beta$ to the fields in S' yields $\underline{E}$ and $\underline{B}$ consistent with equations (15.3), (15.4) and (15.8).

What do these fields look like in frame S ?

ie. how do they depend on R ?

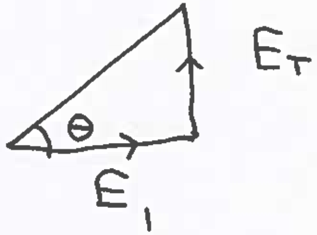


As 3-vector equation we can write

$$\frac{\underline{E}}{c} = \frac{q\gamma}{4\pi\varepsilon_0 c} \frac{\underline{R}}{(R')^3} \qquad (15\cdot 6)$$

NB. $\underline{R} = \left([x^1 - \beta x^0], x^2, x^3 \right)$ is the vector in S from current position of the q (at time x^0) and not the retarded position !?

Magnitude of $\underline{E}$ in S



$$\left((x^2)^2+(x^3)^2\right)^{1/2} = (\ )$$

$$[x^1-\beta x^0] = [\ ]$$

$$|\underline{E}| = \left(E_1^2 + E_T^2\right)^{1/2}$$

$$= \frac{q\gamma}{4\pi\varepsilon_0}\left(\frac{[\ ]^2 + (\ )^2}{\left\{\gamma^2[\ ]^2 + (\ )^2\right\}^3}\right)^{1/2}$$

Note $R^2 = [\ ]^2 + (\ )^2$, $\sin\theta = \dfrac{(\ )}{R}$, $\cos\theta = \dfrac{[\ ]}{R}$

Exercise: Show that we can write:

$$|\underline{E}| = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\frac{1}{\gamma^2}\left(\frac{1}{1-\beta^2\sin^2\theta}\right)^{3/2} \qquad (15.7)$$

$|E|$ at fixed R

- Along the direction of motion $(\sin\theta = 0)$

  $|\underline{E}|$ decreased by $\frac{1}{\gamma^2}$ relative to that for a stationary charge at "current position"!

- Transverse to motion $(\sin\theta = 1)$

  $|\underline{E}|$ <u>increased</u> by factor of γ relative to that for a stationary charge at "current position" $(x' = \beta x'$

Electric Field Produced by a Point Charge Moving with Constant Velocity

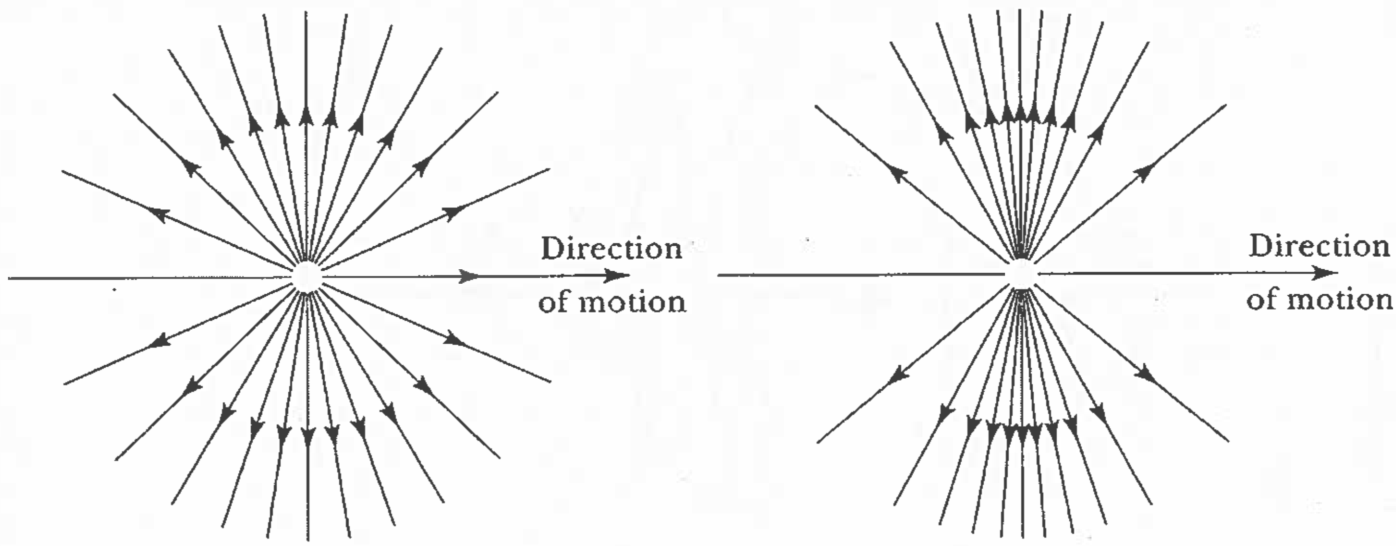


Figure 1: Electric field lines for $\beta = 0.7$ (left) and $\beta = 0.95$ (right) [1].

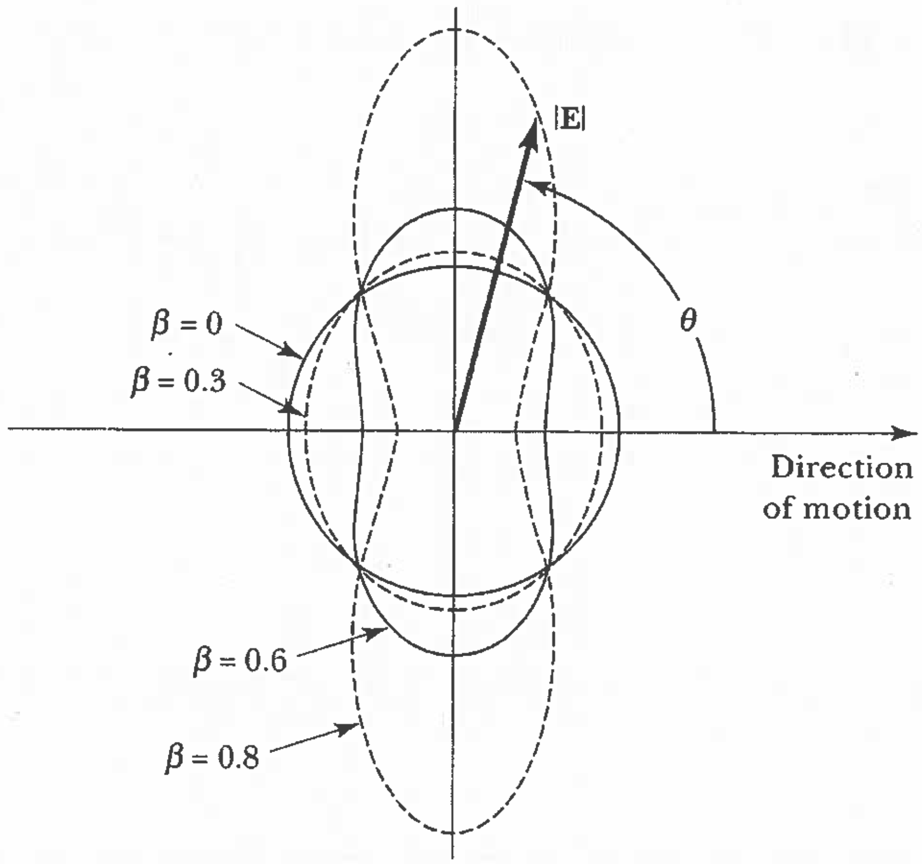


Figure 2: $|\boldsymbol{E}|$ at constant distance, R, from a moving point charge, shown as function of angle to direction of motion, θ, for different values of β [1].

B fields for moving q

$$B_i = -F^{jk} = -\left(\partial^j A^k - \partial^k A^j\right)$$

$$A^1 = \frac{q}{4\pi\varepsilon_0 c}\,\frac{\gamma\beta}{R'}\;, \quad A^2=0,\; A^3=0$$

$$B_1 = 0$$

$$B_2 = -\partial^3 A^1 = \frac{\partial A^1}{\partial x^3} \qquad = -\frac{\beta E_3}{c}$$

$$B_3 = \qquad = \frac{\beta E_2}{c}$$

(15.8)

$$\therefore\; \underline{B} = \frac{1}{c}\left(\underline{\beta}\times\underline{E}\right) \qquad (15.9)$$

Fill in the blanks

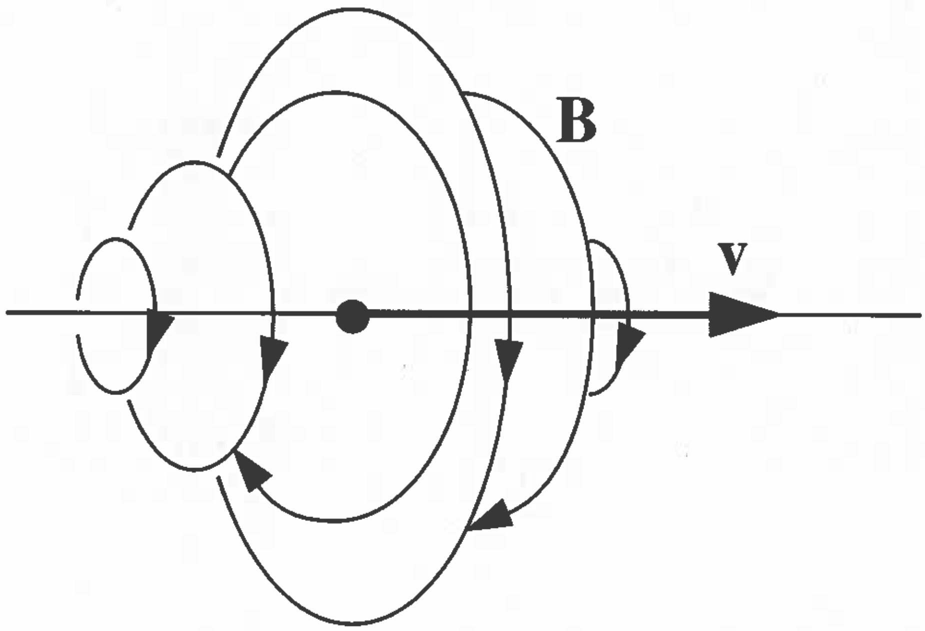


Figure 3: Lines of $\boldsymbol{B}$ relative to direction of motion. [2].

References

[1] *'Classical Electromagnetic Radiation (3rd edition)'*, M.A.Heald and J.B.Marion, Saunders College Publishing.

[2] *'Introduction to Electrodynamics (4th edition)'*, D.J.Griffiths, Pearson Education.