Lecture 15) $\underline{E}$ & $\underline{B}$ fields produced by a moving charged particle

2024

In rest frame S'

$$\frac{\underline{E}}{c} = \frac{q}{4\pi\varepsilon_0 c}\frac{\underline{R}'}{(R')^3}$$

$$\frac{E_1'}{c} = \frac{q}{4\pi\varepsilon_0 c}\frac{x'^1}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\frac{\gamma\left[x^1 - \beta x^0\right]}{(R')^3} \qquad (15.1)$$

$$\left.\begin{aligned} \frac{E_2'}{c} &= \frac{q}{4\pi\varepsilon_0 c}\frac{x'^2}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\frac{x^2}{(R')^3} \\ \frac{E_3'}{c} &= \frac{q}{4\pi\varepsilon_0 c}\frac{x'^3}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\frac{x^3}{(R')^3}\ ? \end{aligned}\right\} \qquad (15.2)$$

Various exercises in this lecture indicated by "?" Fill in the gaps!

Method 1 From expressions for A^μ in frame S (lecture 14) and rules for finding $F^{\mu\nu}$ (ie $\underline{E}, \underline{B}$) from A^μ (lecture 13)

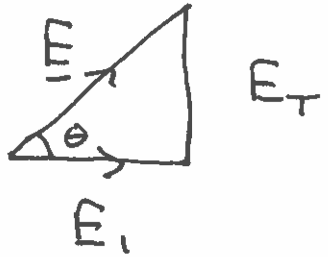
$$\frac{E_1}{c} = F^{10} = \partial^1 A^0 - \partial^0 A^1 = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0}$$

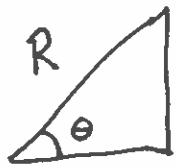
$$= \frac{-q\gamma}{4\pi\varepsilon_0 c}\left[\frac{\left(-\frac{1}{2}\right)2\gamma^2[x^1-\beta x^0]}{(R')^3} + \frac{\beta\left(-\frac{1}{2}\right)2\gamma^2[x^1-\beta x^0](-\beta)}{(R')^3}\right]$$

$$= \frac{q\gamma}{4\pi\varepsilon_0 c}\,\frac{[x^1-\beta x^0]}{(R')^3}\,\underbrace{\gamma^2(1-\beta^2)}_{=1} \qquad (15.3)$$

$$\left.\begin{aligned}
\frac{E_2}{c} &= F^{20} = \partial^2 A^0 - \partial^0 A^2 = -\frac{\partial A^0}{\partial x^2} = \frac{-q\gamma}{4\pi\varepsilon_0 c}\cancel{\gamma}\left(-\frac{1}{2}\right)\frac{2x^2}{(R')^3} = \frac{q\gamma}{4\pi\varepsilon_0 c}\,\frac{x^2}{(R')^3} \\
\frac{E_3}{c} &= F^{30} = \partial^3 A^0 - \partial^0 A^3 = -\frac{\partial A^0}{\partial x^3} = \frac{-q\gamma}{4\pi\varepsilon_0 c}\left(-\frac{1}{2}\right)\frac{2x^3}{(R')^3} = \frac{q\gamma}{4\pi\varepsilon_0 c}\,\frac{x^3}{(R')^3}
\end{aligned}\right\} \quad (15.4)$$

Magnitude of $\underline{E}$ in frame S





$$\left((x^2)^2 + (x^3)^2\right)^{1/2} = (\;)^{1/2}$$

$$[x^1 - \beta x^0] \equiv [\;]$$

$$|\underline{E}| = \left(E_1^2 + E_T^2\right)^{1/2}$$

$$= \frac{q\gamma}{4\pi\varepsilon_0}\left(\frac{[\;]^2 + (\;)^2}{\left\{\gamma^2[\;]^2 + (\;)^2\right\}^3}\right)^{1/2} \qquad \text{using (15.3) \& (15.4)}$$

note: $R^2 = [\;]^2 + (\;)^2$, $\sin\theta = \dfrac{(\;)}{R}$, $\cos\theta = \dfrac{[\;]}{R}$

Exercise: Show that we can write

$$|\underline{E}| = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\frac{1}{\gamma^2}\left(\frac{1}{1-\beta^2\sin^2\theta}\right)^{3/2} \qquad (15.7)$$

Using the results leading up to equation (15.7)

$$\left(\frac{[\;]^2 + (x^2)^2}{\{\gamma^2[\;]^2 + (x^2)^2\}^3}\right)^{1/2} = \left(\frac{R^2}{\{\gamma^2 R^2 \cos^2\theta + R^2\sin^2\theta\}^3}\right)^{1/2}$$

$$= \frac{R}{(\gamma^2R^2)^{3/2}}\left(\frac{1}{1-\sin^2\theta + (1-\beta^2)\sin^2\theta}\right)^{3/2}$$

$$= \frac{1}{\gamma^3R^2}\left(\frac{1}{1-\beta^2\sin^2\theta}\right)^{3/2},$$

which gives (15.7) when substituted into the expression for $|E|$

B fields for moving charge

$$B_i = -F^{jk} = -\left(\partial^j A^k - \partial^k A^j\right)$$

$$A^1 = \frac{q}{4\pi\varepsilon_0 c}\,\frac{\gamma\beta}{R'}\,, \qquad A^2 = A^3 = 0$$

$$B_1 = 0$$

$$B_2 = -\partial^3 A^1 = \frac{\partial A^1}{\partial x^3} = \frac{q\gamma\beta}{4\pi\varepsilon_0 c}\,\frac{-\left(\frac{1}{2}\right)2x^3}{(R')^3} = -\frac{\beta E_3}{c}$$

$$B_3 = \partial^2 A^1 = -\frac{\partial A^1}{\partial x^2} = \frac{-q\gamma\beta}{4\pi\varepsilon_0 c}\,\frac{\left(-\frac{1}{2}\right)2x^2}{(R')^3} = \frac{\beta E_2}{c}$$

(15.8)

(using equations (15.4))

$$\underline{B} = \frac{1}{c}\left(\underline{\beta} \times \underline{E}\right) \qquad (15.9)$$

Method 2) Exercise for you.

Use transformation equations, $F'^{\mu\nu}$, on (15.1) & (15.2) to obtain same results for $\underline{E}$ and $\underline{B}$ (15.3), (15.4), (15.9)

Note: This material covered in 2nd half of Mini-Lecture 15(b).

Apply the general transformations to the case of our point charge q.

Note: We need the inverse transformations S in terms of S'

$\therefore$ Set $\beta \to -\beta$ in the equations we have just worked out.

Since $\underline{B}' = 0$ we can write:

$$\frac{E_1}{c} = \frac{E_1'}{c} = \frac{q}{4\pi\varepsilon_0 c}\,\frac{\gamma[x^1 - \beta x^0]}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c}\,\frac{\gamma[x^1 - \beta x^0]}{\left[(\gamma[x^1 - \beta x^0])^2 + (x^2)^2 + (x^3)^2\right]^{3/2}}$$

$$\frac{E_2}{c} = \gamma \frac{E_2'}{c} = \frac{q}{4\pi\varepsilon_0 c}\,\gamma\,\frac{x^2}{(R')^3}$$

$$\frac{E_3}{c} = \gamma \frac{E_3'}{c} = \frac{q}{4\pi\varepsilon_0 c}\,\gamma\,\frac{x^3}{(R')}$$

consistent with (15.3) & (15.4)

$F^{\mu\nu}$ transforms as a 2nd rank tensor:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta} \qquad\qquad F^{\mu\nu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\alpha}\left(\Lambda^{-1}\right)^{\nu}{}_{\beta}F'^{\alpha\beta}$$

or, alternatively,

$$F' = \Lambda F \Lambda^{T} \qquad\qquad F = \left(\Lambda^{-1}\right)F'\left(\Lambda^{-1}\right)^{T},$$

from which it can be shown that:

$$\frac{E_1'}{c} = \frac{E_1}{c} \qquad B_1' = B_1 \qquad\qquad \frac{E_1}{c} = \frac{E_1'}{c} \qquad B_1 = B_1'$$

$$\frac{E_2'}{c} = \gamma\left(\frac{E_2}{c} - \beta B_3\right) \qquad B_2' = \gamma\left(B_2 + \beta\frac{E_3}{c}\right) \qquad\qquad \frac{E_2}{c} = \gamma\left(\frac{E_2'}{c} + \beta B_3'\right) \qquad B_2 = \gamma\left(B_2' - \beta\frac{E_3'}{c}\right)$$

$$\frac{E_3'}{c} = \gamma\left(\frac{E_3}{c} + \beta B_2\right) \qquad B_3' = \gamma\left(B_3 - \beta\frac{E_2}{c}\right) \qquad\qquad \frac{E_3}{c} = \gamma\left(\frac{E_3'}{c} - \beta B_2'\right) \qquad B_3 = \gamma\left(B_3' + \beta\frac{E_2'}{c}\right)$$

Notes:

1. The inverse transformation for the fields (from S' to S) may be obtained by setting $\beta \longrightarrow -\beta$ in the above equations.

2. The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are 3-D vectors that are not components of a 4-vector. For the purposes of this course, I make no distinction between "upper" and "lower" indices for such 3-D vectors.

Magnetic fields in frame S

$$
\left.\begin{aligned}
B_1 &= B_1' = 0 \\
B_2 &= \gamma\left(B_2' - \beta \frac{E_3'}{c}\right) = -\gamma\beta \frac{E_3'}{c} = -\beta \frac{E_3}{c} \\
B_3 &= \gamma\left(B_3' + \beta \frac{E_2'}{c}\right) = \gamma\beta \frac{E_2'}{c} = \beta \frac{E_2}{c}
\end{aligned}\right\} \text{as in } (15.8)
$$

again $\underline{B} = \frac{1}{c}\left(\underline{\beta} \times \underline{E}\right)$ (as in 15.9)