Lecture 16 More Electrodynamics in Lorentz-Covariant Notation

The inhomogeneous Maxwell equations can be written as

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 j^\nu} \qquad (16.1)$$

Let's consider $\nu = 0$ component

$$\partial_\mu F^{\mu\nu} = 0 + \frac{\partial}{\partial x^1}\left(\frac{E_1}{c}\right) + \frac{\partial}{\partial x^2}\left(\frac{E_2}{c}\right) + \frac{\partial}{\partial x^3}\left(\frac{E_3}{c}\right)$$

$$= \frac{1}{c}\nabla \cdot \underline{E} = \mu_0 j^0 = \mu_0 c \rho$$

$$\therefore \quad \boxed{\nabla \cdot \underline{E} = \rho/\varepsilon_0}$$

Reminder

$$
\Lambda^{\delta}{}_{\gamma} = \begin{array}{c|c} & \gamma \rightarrow \\ \delta \downarrow & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \end{array}
$$

$$
\Lambda^{\delta}{}_{\gamma} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)
$$

$$
F^{\alpha\beta} = \begin{array}{c} \alpha \downarrow \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \beta \rightarrow \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{cccc} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{array} \right) \end{array}
$$

$$
\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla \right)
$$

$$
J^{\mu} = \rho\left(c, \underline{v} \right)
$$

For $\nu = 1$

$$\partial_\mu F^{\mu 1} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{E_1}{c}\right) + 0 + \frac{\partial}{\partial x^2}\left(B_3\right) + \frac{\partial}{\partial x^3}\left(-B_2\right)$$

$$= \left[-\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} + \nabla \times \underline{B}\right]_1 = \mu_0 j^1$$

$$\therefore \quad \boxed{\nabla \times \underline{B} = \mu_0 \underline{j} + \varepsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}}$$

$\because$ if we can demonstrate for one cartesian coordinate must be true in general.

Exercise : verify explicitly for $\nu = 2, 3$.

Note (16.1) is closely related to the wave equation for A^μ

$$\mu_0 j^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right)$$

$$= \Box^2 A^\nu - \partial^\nu \underbrace{\left(\partial_\mu A^\mu \right)}_{=0 \text{ in Lorentz gauge}}.$$

An exercise for you

Show that

$$\boxed{\partial^\mu F^{\nu\lambda} + \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} = 0}$$

correspond to the homogeneous Maxwell Equations

Hint: try using $\mu, \nu, \lambda = 1, 2, 3$

and $\quad \ldots = 0, 1, 2$

The interaction of point charged particles with the fields

$$\boxed{\frac{dp^\mu}{d\tau} = q F^{\mu\nu} u_\nu}$$

↑

proper time

preserves 4-vector nature of L.H.S.

Reminder

$$
\Lambda^{\delta}{}_{\gamma} =
\begin{array}{c}
\gamma \rightarrow \\
\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} \delta \downarrow \\ 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left(\begin{array}{cc|cc}
\gamma & -\gamma\beta & 0 & 0 \\
-\gamma\beta & \gamma & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
\end{array}
\end{array}
$$

$$
F^{\alpha\beta} =
\begin{array}{c}
\beta \rightarrow \\
\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} \alpha \downarrow \\ 0 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{pmatrix}
0 & -E_1/c & -E_2/c & -E_3/c \\
E_1/c & 0 & -B_3 & B_2 \\
E_2/c & B_3 & 0 & -B_1 \\
E_3/c & -B_2 & B_1 & 0
\end{pmatrix}
\end{array}
\end{array}
$$

$$p^{\mu} = \left(\frac{E}{c}, \underline{p}\right), \quad \text{where } \underline{p} = \gamma m \underline{v}$$

$$u_{\mu} = \gamma(c, -\underline{v})$$

Start with $\mu = 0$ of R.H.S.

$$q F^{0\nu} u_\nu = q\left[0 + \left(-\frac{E_1}{c}\right)\left(-\gamma v_1\right) + \left(-\frac{E_2}{c}\right)\left(-\gamma v_2\right) + \left(-\frac{E_3}{c}\right)\left(-\gamma v_3\right) \right]$$

$$= q\frac{\gamma \underline{v}\cdot\underline{E}}{c} \qquad (16\cdot 2)$$

$\mu = 1$

$$q F^{1\nu} u_\nu = q\left[\frac{E_1}{c}\gamma c + 0 + \left(-B_3\right)\left(-\gamma v_2\right) + B_2\left(-\gamma v_3\right) \right]$$

$$= q\gamma\left[\underline{E} + \underline{v} \times \underline{B} \right]_1 \qquad (16\cdot 3)$$

LHS. $\quad dt = \gamma \, d\tau$

<u>$\mu = 0$ term</u>

$$\frac{dp^0}{d\tau} = \frac{\gamma \, dp^0}{dt} = \gamma \frac{d}{dt}\left(\frac{\mathcal{E}}{c}\right) = q\gamma \frac{\underline{v} \cdot \underline{E}}{c} \qquad \text{using } (16.2)$$

$$\boxed{\frac{d\mathcal{E}}{dt} = q \, \underline{v} \cdot \underline{E}}$$

rate of work done by
E field on charge

$\mu = 1$ term on LHS.

$$\frac{dp^1}{d\tau} = \gamma \frac{dp^1}{dt} = q F^{1\nu} u_\nu = q\gamma\left[\underline{E} + \underline{v} \times \underline{B}\right],$$

(using 16·3)

$$\therefore \quad \boxed{\frac{d\underline{p}}{dt} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)}$$

Lorentz Force Law

where $\underline{p} = \gamma m \underline{v}$