For $\nu = 1$

$$\partial_\mu F^{\mu 1} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{E_1}{c}\right) + 0 + \frac{\partial}{\partial x^2}\left(B_3\right) + \frac{\partial}{\partial x^3}\left(-B_2\right)$$

$$= \left[-\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} + \nabla \times \underline{B}\right]_1$$

$$= \mu_0 j^1$$

$$\therefore \quad \nabla \times \underline{B} = \mu_0 \underline{j} + \varepsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t}$$

$\because$ if we demonstrate for one cartesian coordinate it must be true in general!

Exercise : just for practice, repeat the above for 2, 3 components

For $\nu = 2$

$$\partial_\mu F^{\mu 2} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{E_2}{c}\right) + \frac{\partial}{\partial x^1}\left(-B_3\right) + 0 + \frac{\partial (B_1)}{\partial x^3}$$

$$= \left[-\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} + \nabla \times \underline{B}\right]_2 = \mu_0 j^2$$

For $\nu = 3$

$$\partial_\mu F^{\mu 3} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{E_3}{c}\right) + \frac{\partial}{\partial x^1}\left(B_2\right) + \frac{\partial}{\partial x^2}\left(-B_1\right)$$

$$= \left[-\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} + \nabla \times \underline{B}\right]_3 = \mu_0 j^3$$

as required.

An exercise for you

Show that

$$\boxed{\partial^\mu F^{\nu\lambda} + \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} = 0}$$

corresponds to the homogeneous Maxwell equations.

Note the cyclic permutation of indices.

Hint: Use $\mu, \nu, \lambda = 1, 2, 3$

and then $0, 1, 2$.

For $\mu, \nu, \lambda = 1, 2, 3$

$$0 = \partial^1 F^{23} + \partial^3 F^{12} + \partial^2 F^{31}$$

$$= -\left[\frac{\partial}{\partial x^1}(-B_1) + \frac{\partial}{\partial x^3}(-B_3) + \frac{\partial}{\partial x^2}(-B_2)\right] = \nabla . \underline{B}$$

from $\partial^i = \frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x^i}$

$$\therefore \quad \nabla . \underline{B} = 0$$

For $\mu, \nu, \lambda = 0, 1, 2$

$$0 = \partial^0 F^{12} + \partial^2 F^{01} + \partial^1 F^{20}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(-B_3\right) + \left(-\frac{\partial}{\partial x^2}\right)\left(-\frac{E_1}{c}\right) + \left(-\frac{\partial}{\partial x^1}\right)\left(\frac{E_2}{c}\right)$$

$$= -\frac{1}{c}\left[\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E}\right]_3$$

Similar results can be obtained for

$\mu, \nu, \lambda = 0, 2, 3$ and $0, 3, 1$,

thus demonstrating that

$$\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E} = 0$$

More generally, we could write

$$0 = \partial^0 F^{ij} + \partial^j F^{0i} + \partial^i F^{j0}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(-\varepsilon_{ijk} B_k\right) + \left(-\frac{\partial}{\partial x^j}\right)\left(-\frac{E_i}{c}\right) + \left(-\frac{\partial}{\partial x^i}\right)\left(\frac{E_j}{c}\right)$$

$$= \frac{-\varepsilon_{ijk}}{c}\frac{\partial B_k}{\partial t} - \frac{\varepsilon_{ijk}}{c}\left[\nabla \times \underline{E}\right]_k$$

$$= -\frac{\varepsilon_{ijk}}{c}\left[\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E}\right]_k$$

which shows that $\left[\frac{\partial \underline{B}}{\partial t} + \nabla \times \underline{E}\right]_k = 0 \quad \forall k$

Interaction of charged particles with the fields

Lets consider the rate of change of 4-momentum with proper time

$$\boxed{\frac{dp^\mu}{d\tau} = q\,F^{\mu\nu}\,u_\nu}$$

Pretty much has to depend on these 4-vectors

$$= q\begin{bmatrix} 0 & -\frac{E_1}{c} & -\frac{E_2}{c} & -\frac{E_3}{c} \\ \frac{E_1}{c} & 0 & -B_3 & B_2 \\ \frac{E_2}{c} & B_3 & 0 & -B_1 \\ \frac{E_3}{c} & -B_2 & B_1 & 0 \end{bmatrix}\begin{bmatrix} \gamma c \\ -\gamma u_1 \\ -\gamma u_2 \\ -\gamma u_3 \end{bmatrix} = q\gamma\begin{bmatrix} \frac{E_1 u_1}{c} + \frac{E_2 u_2}{c} + \frac{E_3 u_3}{c} \\ E_1 + B_3 u_2 - B_2 u_3 \\ E_2 - B_3 u_1 + B_1 u_3 \\ E_3 + B_2 u_1 - B_1 u_3 \end{bmatrix}$$

$$= q\gamma\left(\frac{\underline{u}.\underline{E}}{c},\ \underline{E} + \underline{u}\times\underline{B}\right)$$

N.B. u_i here are components of 3D vector $\underline{u}$. No distinction between upper or lower indices

For $\mu = 2$

$$q F^{2\nu} u_\nu = q\left[\frac{E_2}{c}(\gamma c) + B_3(-\gamma u_1) + 0 + (-B_1)(-\gamma u_3) \right]$$

$$= \gamma q \left[\underline{E} + \underline{u} \times \underline{B} \right]_2$$

For $\mu = 3$

$$q F^{3\nu} = q\left[\frac{E_3}{c}(\gamma c) + (-B_2)(-\gamma u_1) + B_1(-\gamma u_2) + 0 \right]$$

$$= \gamma q \left[\underline{E} + \underline{u} \times \underline{B} \right]_3$$

General result for $\mu \neq 0$ or $\mu = i$, $i = 1, 2, 3$

$$\frac{dp^i}{d\tau} = \gamma \frac{dp^i}{dt} = q F^{i\nu} u_\nu$$

Here $\nu \neq 0$

$\nu = j \neq i$

$$= q \left\{ F^{io} u_o + \varepsilon_{ijk} (-B_k)(-\gamma u_j) \right\}$$

u_j is component of 3D vector $\underline{u}$

$$= q \left\{ \frac{E_i}{c} (\gamma c) + \gamma \left[\underline{u} \times \underline{B} \right]_i \right\}$$

$$= \gamma q \left[\underline{E} + \underline{u} \times \underline{B} \right]_i \quad \forall i$$

$$\therefore \quad \frac{d\underline{p}}{dt} = q \left(\underline{E} + \underline{u} \times \underline{B} \right)$$ as required