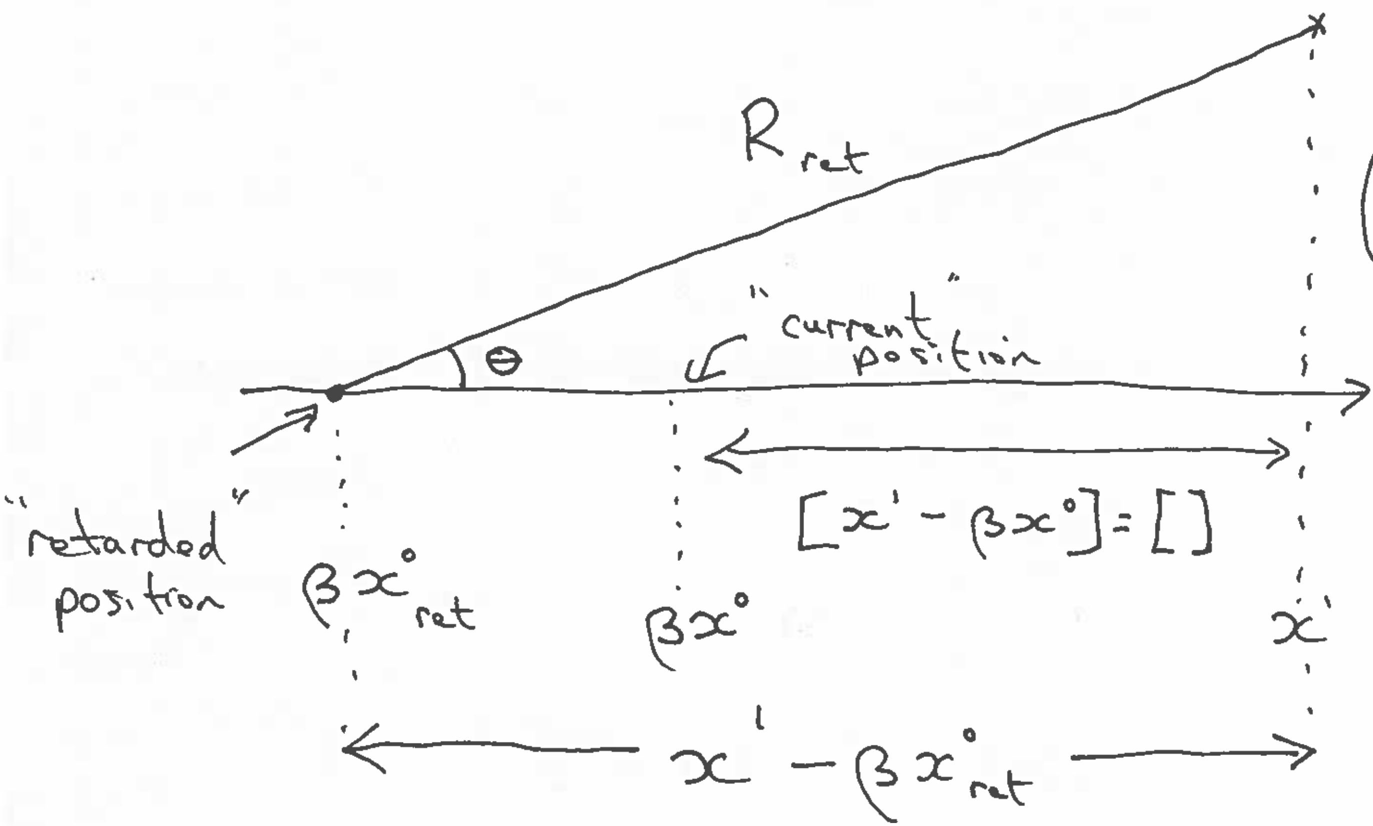
Lecture 19) Accelerating point charge – vector calculus approach

Part 1) Re-express A^μ in terms of $[R, \theta, \beta]_{ret}$



$$\left((x^2)^2 + (x^3)^2\right)^{1/2} = (\;)$$

$\underline{\beta} \rightarrow \underline{x}^1$

Potentials in frame S (Reminder: From Lecture 14)

L.T. $\frac{V}{c} = A^0 = \gamma(A'^0 + \beta A'^1) = \gamma A'^0$ (Note: inverse Lorentz Transformation.)

$$\therefore \quad A^0 = \frac{q}{4\pi\varepsilon_0 c}\gamma\frac{1}{\left[\left(\gamma[x^1-\beta x^0]\right)^2+(x^2)^2+(x^3)^2\right]^{1/2}}$$

$$A^1 = \gamma(\beta A'^0 + A'^1) = \gamma\beta A'^0 = \beta A^0$$

$$= \frac{q}{4\pi\varepsilon_0 c}\gamma\beta\frac{1}{\left[\left(\gamma[x^1-\beta x^0]\right)^2+(x^2)^2+(x^3)^2\right]^{1/2}}$$

$$A^2 = A^3 = 0$$

$\underline{A} = \underline{\beta} A^0$

Exercise for you: Cross-check that $A'^{\mu}A'_{\mu} = A^{\mu}A_{\mu}$.

From Lecture 14

$$A^{0} = \frac{q}{4\pi\varepsilon_0 c}\,\gamma\,\frac{1}{\left[(\gamma[\,])^{2} + (\,)^{2}\right]^{1/2}} = \frac{q}{4\pi\varepsilon_0 c}\,\frac{1}{\left[[\,]^{2} + (1-\beta^{2})(\,)^{2}\right]^{1/2}} \qquad (19.1)$$

Some useful relations from diagram

$$(\,) = R_{ret}\sin\theta \qquad (19.2)$$

$$x^{1} - \beta x^{0}_{ret} = R_{ret}\cos\theta \qquad (19.3)$$

From $t_{ret} = t - R_{ret}/c \qquad (19.4)$

$$x^{0}_{ret} + R_{ret} = x^{0}$$

$$[\,] = x^{1} - \beta\left(x^{0}_{ret} + R_{ret}\right) = R_{ret}\left(\cos\theta - \beta\right) \qquad (19.5)$$

(using 19.3)

Exercise for you: show that

$$\left[\,[\;]^2 + (1-\beta^2)(\;)^2\right] = \left[R_{ret}(1-\beta\cos\theta)\right]^2$$

$$= \left[R_{ret}(1-\underline{\beta}\cdot\hat{R}_{ret})\right]^2 \qquad (19.6)$$

Substitute (19.6) into (19.1)

$$A^0 = \frac{q}{4\pi\varepsilon_0 c} \frac{1}{\left[R\left(1 - \underline{\beta} \cdot \hat{\underline{R}}\right)\right]_{ret}} \qquad (19.7)$$

where $[\;]_{ret}$ indicates that $\underline{R}$ & $\underline{\beta}$ are at the retarded space-time point, x^{μ}_{ret}.

completely equivalent to the results from Lecture 14
– just re-expressed in terms of R_{ret}, $\underline{\beta}_{ret}$
which is what we need for accelerating charges.

From lecture 14)

$$\underline{A} = \underline{\beta} A^0$$

$$\underline{A} = \frac{q}{4\pi\varepsilon_0 c} \left[\frac{\underline{\beta}}{R(1 - \underline{\beta} \cdot \hat{\underline{R}})} \right]_{ret} \qquad (19.8)$$

(19.7) & (19.8) are the <u>Liénard - Wiechert Potentials</u> for a moving point charge

Part 2) Use vector calculus to find $\underline{E}$, $\underline{B}$ produced by an accelerating charge

Consider the case $\beta \to 0$

Just for brevity drop "ret" from R_{ret}, β_{ret} and put back in at the end.

$$\underline{E} = -\nabla V - \frac{\partial \underline{A}}{\partial t} \qquad (19.9)$$

Using (19.7)

$$-\nabla V - c\nabla A^{0} = \frac{-q}{4\pi\varepsilon_0} \frac{-1}{[R(1-\underline{\beta}.\hat{\underline{R}})]^2} \left(\nabla R - \nabla(\underline{\beta}.\underline{R}) \right) \qquad (19.10)$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\boxed{\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Exercise: show that $\nabla R = \underline{\hat{R}}$

Using V.I. (4)

$$-\nabla(\underline{\beta}\cdot\underline{R}) = \underbrace{-\underline{\beta}\times(\nabla\times\underline{R})}_{\to 0 \text{ as } \beta\to 0} - \underline{R}\times(\nabla\times\underline{\beta}) - \underbrace{(\underline{\beta}\cdot\nabla)\underline{R}}_{\to 0 \text{ as } \beta\to 0} - (\underline{R}\cdot\nabla)\underline{\beta} \quad (19.11)$$

$$\nabla\times\underline{\beta} = \varepsilon_{ijk}\frac{\partial}{\partial x^j}\beta^k = \varepsilon_{ijk}\frac{\partial t_{ret}}{\partial x^j}\frac{\partial\beta^k}{\partial t_{ret}} = \nabla t_{ret}\times\dot{\underline{\beta}} \quad (19.2)$$

where we have defined $\dot{\underline{\beta}} = \frac{\partial\underline{\beta}}{\partial t_{ret}}$

$$t_{ret} = t - \frac{R}{c} \underset{\beta\to 0}{\Longrightarrow} \nabla t_{ret} = -\frac{\nabla R}{c} = -\frac{\underline{\hat{R}}}{c} \quad (19.13)$$