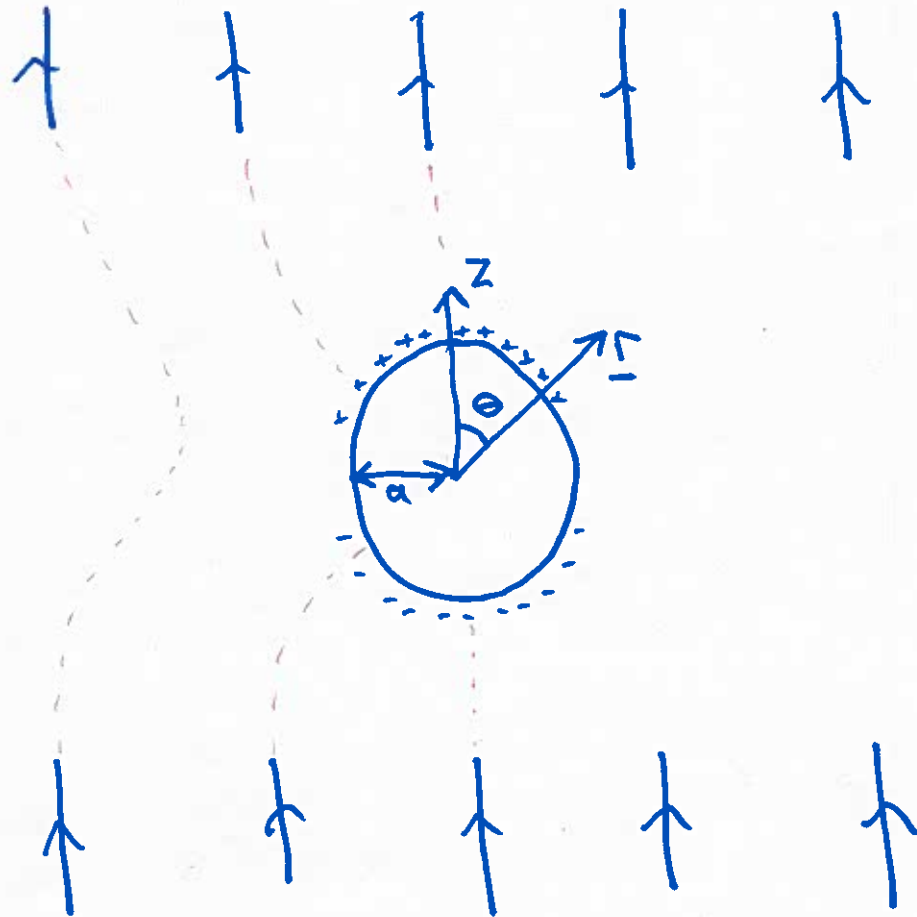


# Lecture 4) Example of a Laplace's Equation in Spherical

## Polar Coordinates



At surface of conductor  
 $V = \text{constant} = 0$

Field at surface has to  
be radial

$\therefore$  rotational symmetry  
we can ignore  $\phi$

At surface:  $r = a$

$$\underline{E}(r, \theta, \phi) = \underline{E}(a, \theta) \hat{r}$$

$$\underline{E} \text{ inside surface} = 0$$

Large distances  $r \rightarrow \infty$

$$V = -E_0 z = -E_0 r \cos \theta$$

General Formalism (4.1)

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

Separation of Variables

$$V(r, \theta) = R(r) \Theta(\theta) \quad (4.2)$$

Exercise for you : combine (4.1) & (4.2) to show

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0 \quad (4.3)$$

constants :  $l(l+1)$

$-l(l+1)$

## VECTOR DERIVATIVES

---

**Cartesian.**  $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

*Curl:*  $\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

*Laplacian:*  $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

**Spherical.**  $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

*Curl:*  $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$

*Laplacian:*  $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

**Cylindrical.**  $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

*Curl:*  $\nabla \times \mathbf{v} = \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$

*Laplacian:*  $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

State general solution

$$V(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Exercise  
for you:

Verify that

solution for  $R(r)$

solution for  $\Theta(\theta)$

where  $P_l(\cos \theta)$  are the Legendre Polynomials

$$P_0(\cos \theta) = 1$$

$$P_1(\cos \theta) = \cos(\theta)$$

$$P_2 = (3 \cos^2 \theta - 1)/2$$

Return to our specific case and apply the boundary conditions

$$\bullet r \rightarrow \infty : V = -E_0 r \cos \theta = -E_0 r P_1(\cos \theta)$$

$\therefore P_l$  are orthogonal

$$\therefore A_l = 0 \text{ except for } A_1 r \cos \theta = -E_0 r \cos \theta$$

(terms in  $\frac{1}{r^{l+1}} \rightarrow 0$  as  $r \rightarrow \infty$ )

$$\therefore V = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

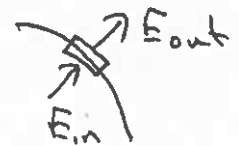
$$\bullet r = a \quad V = 0 \Rightarrow B_l = 0 \text{ except for } l = 1$$

$$0 = -E_0 a \cos \theta + \frac{B_1}{a^2} \cos \theta \quad \therefore B_1 = E_0 a^3$$

$$\therefore V(r, \theta) = -E_0 \left( r - \frac{a^3}{r^2} \right) \cos \theta \quad \left( \text{valid for } r > a \right)$$

Exercise for you

$$1) \quad \underline{E} = -\nabla V = - \left( \frac{\partial V}{\partial r} \underline{\hat{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \underline{\hat{\theta}} \right)$$



2) Apply Gauss' Law at surface of sphere to work out surface charge density

$$\sigma(\theta) =$$

## VECTOR DERIVATIVES

**Cartesian.**  $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

**Gradient:**  $\nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

**Divergence:**  $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

**Curl:**  $\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

**Laplacian:**  $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

**Spherical.**  $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

**Gradient:**  $\nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$

**Divergence:**  $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

**Curl:**  $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$

**Laplacian:**  $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

**Cylindrical.**  $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

**Gradient:**  $\nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

**Divergence:**  $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

**Curl:**  $\nabla \times \mathbf{v} = \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$

**Laplacian:**  $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

## Note

For systems without symmetry in  $\phi$  solutions in terms of

"Spherical Harmonics"  $Y_l^m(\theta, \phi)$

"Extra curricular" reading see, e.g.,

Heald and Marion : Section 3.4

Jackson : Chapter 3.5