General formalism for 2D (r, θ) problems

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0 \quad (4.1)$$

Separation of variables:

$$V(r, \theta) = \mathbb{R}(r)\,\Theta(\theta) \quad (4.2)$$

Exercise for you: combine (4.1) & (4.2) to obtain

$$\underbrace{\frac{1}{\mathbb{R}}\frac{d}{dr}\left(r^2 \frac{d\mathbb{R}}{dr}\right)}_{\ell(\ell+1)} + \underbrace{\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)}_{-\ell(\ell+1)} = 0 \quad (4.3)$$

constants

State general solution $V(r,\theta)$

$$V(r,\theta) = \sum_{l=0}^{\infty} \underbrace{\left(A_l r^l + \frac{B_l}{r^{l+1}} \right)}_{\text{Solution to equation for } R(r)} \underbrace{P_l(\cos\theta)}_{\Theta(\theta)}$$

Exercise for you: Verify that

$P_l(\cos\theta)$ are the Legendre Polynomials

$$P_0(\cos\theta) = 1$$

$$P_1 = \cos\theta$$

$$P_2 = (3\cos^2\theta - 1)/2$$

Exercises

1) $$\underline{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\,\hat{\underline{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\underline{\theta}}\right)$$

$\hat{\underline{r}} \sim \underline{E}$

2 Apply gauss's law at surface of sphere to work out $\sigma(\theta)$