

Verify general solution in $\mathbb{R}$

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \ell(\ell+1) R \quad \text{is the differential equation in } \mathbb{R}$$

$$\frac{d}{dr} \left(r^2 \frac{d}{dr} \left[A r^\ell + \frac{B}{r^{\ell+1}} \right] \right) = \frac{d}{dr} \left(r^2 \left[\ell A r^{\ell-1} - (\ell+1) \frac{B}{r^{\ell+2}} \right] \right)$$

$$= \frac{d}{dr} \left(\ell A r^{\ell+1} - (\ell+1) \frac{B}{r} \right) = \ell(\ell+1) A r^\ell + \ell(\ell+1) \frac{B}{r^{\ell+1}}$$

$$= \ell(\ell+1) R, \quad \text{as required.}$$

Verify solutions to the differential equation for Θ

$$\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = -l(l+1) \sin \theta \Theta \quad \text{is the relevant equation}$$

For $P_0 = 1$

$$\frac{d}{d\theta} \left(\sin \theta \frac{d[1]}{d\theta} \right) = 0, \quad \text{as required for } l=0$$

For $P_1 = \cos \theta$

$$\frac{d}{d\theta} \left(\sin \theta \frac{d[\cos \theta]}{d\theta} \right) = - \frac{d}{d\theta} \left(\sin^2 \theta \right) = -2 \sin \theta \cos \theta$$

$$= -1(1+1) \sin \theta P_1, \quad \text{as required for } l=1$$

$$\text{For } P_2 = (3 \cos^2 \theta - 1)/2$$

$$\frac{d}{d\theta} \left(\sin \theta \frac{d}{d\theta} \left[\frac{3 \cos^2 \theta - 1}{2} \right] \right) = \frac{d}{d\theta} \left(\sin \theta [-3 \cos \theta \sin \theta] \right)$$

$$= -3 \frac{d}{d\theta} [\sin^2 \theta \cos \theta] = -3 [2 \sin \theta \cos^2 \theta - \sin^3 \theta]$$

$$= -3 \sin \theta (3 \cos^2 \theta - 1) \quad \left\{ \text{since } \sin^3 \theta = \sin \theta (1 - \cos^2 \theta) \right.$$

$$= -2(2+1) \sin \theta P_2, \text{ as required for } l=2$$

$$V(r, \theta) = -E_0 \left(r - \frac{a^3}{r^2} \right) \cos \theta$$

$\uparrow$ applied field $\uparrow$ induced charge on sphere.

Electric Field

$$\underline{E} = -\nabla V = - \left(\frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} \right)$$

$$= E_0 \left(1 + 2 \frac{a^3}{r^3} \right) \cos \theta \hat{r} - E_0 \left(1 - \frac{a^3}{r^3} \right) \sin \theta \hat{\theta}$$

At $r=a$

$$E_{\theta} = 0 \quad \checkmark$$

$$E_r = 3E_0 \cos \theta = \frac{\sigma(\theta)}{\epsilon_0} \quad \text{from Gauss's Law}$$

$$\therefore \sigma(\theta) = 3\epsilon_0 E_0 \cos \theta$$

