Lecture 7 Topics in Magnetostatics

A) General derivation of $\nabla \times \underline{B} = \mu_0 \underline{j}$

A slight diversion back to Electrostatics

Consider a point charge at $\underline{r}' = \underline{a}$

$$q = \int_V \rho(\underline{r}')\, d\tau'$$

requires
$$\rho(\underline{r}') = q\,\delta^3(\underline{r}' - \underline{a}) \qquad (7.1)$$

$$\frac{\rho(\underline{r}')}{\varepsilon_0} = -\nabla^2 V(\underline{r}) = \frac{-q}{4\pi\varepsilon_0}\nabla^2\left(\frac{1}{R}\right) = \nabla\cdot\underline{E}(\underline{r})$$

Point particle at $\underline{r}'$

$$= \frac{q}{4\pi\varepsilon_0}\nabla\cdot\left(\frac{\hat{R}}{R^2}\right) \qquad (7.2)$$

Compare (7.1) & (7.2)

$$-\nabla^2\left(\frac{1}{R}\right) = \nabla\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\delta^3(\underline{R}) = 4\pi\delta^3(\underline{r}-\underline{r}') \qquad (7.3)$$

Note: ρ, $\nabla^2 V$, $\nabla\cdot\underline{E}$ are zero everywhere except at the location of the point charge.

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Back to Magnetostatics

$$\nabla \times \underline{B} = \nabla \times (\nabla \times \underline{A}) = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A} \qquad [V.I.11]$$

Gauge freedom $\underline{A} \Rightarrow \underline{A} + \nabla f$ no change in $\underline{B}$ [V.I.10]

Choose ∇f such that $\nabla \cdot \underline{A} = 0$ "Coulomb gauge"

Then

$$\nabla \times \underline{B}(\underline{r}) = -\nabla^2 A(\underline{r}) = \frac{-\mu_0}{4\pi} \int_v \nabla^2 \left\{ \frac{\underline{j}(\underline{r}')}{R} \right\} d\tau'$$

$$= \frac{-\mu_0}{4\pi} \int_v \underline{j}(\underline{r}') \, \nabla^2 \left(\frac{1}{R} \right) d\tau$$

$\because$ $\underline{j}(\underline{r}')$ does not depend explicitly on unprimed coordinates

$$\nabla \times \underline{B}(\underline{r}) = -\frac{\mu_0}{4\pi} \int_V \underline{j}(\underline{r}') \left[-4\pi \, \delta(\underline{r} - \underline{r}') \right] d\tau'$$

[using 7.3]

$$= \mu_0 \, \underline{j}(\underline{r})$$

Since $\underline{B}$, $\underline{A}$, $\underline{j}$ are all evaluated at $\underline{r}$ we can write:

$$\nabla \times \underline{B} = -\nabla^2 \underline{A} = \mu_0 \underline{j}$$

<u>Notes:</u> 1) More general than in Lecture 6.

2) We have obtained a "vector version" of Poissons Equation or 3 scalar equations:

$$\nabla^2 A_i = \mu_0 j_i$$

3) $\nabla \times \underline{B}$, $\nabla^2 \underline{A}$ depend only on $\underline{j}$ at the same point in space!

Topic B

Is our definition

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\underline{j}(\underline{r}')}{R}\, d\tau'$$

consistent with the choice $\nabla \cdot \underline{A} = 0$

Two mathematical techniques

1) Define a new symbol

$$\nabla_{r'} \equiv \hat{\underline{x}}\frac{\partial}{\partial x'} + \hat{\underline{y}}\frac{\partial}{\partial y'} + \hat{\underline{z}}\frac{\partial}{\partial z'}$$ w.r.t. the primed coordinates

$$\nabla_{r'} f(\underline{r}) = 0$$

$$\nabla_{r'} f(R) = \nabla_{r'} f(\underline{r}-\underline{r}') = -\nabla f(\underline{r}-\underline{r}') \qquad (7.4)$$

2) "Integratation by parts" in vector calculus

eg. [V.I. 5]

$$\int_V f(\nabla\cdot\underline{A})\,d\tau' + \int_V \underline{A}\cdot(\nabla f)\,d\tau' = \int_V \nabla\cdot(f\underline{A})\,d\tau'$$

$$= \int_S f\underline{A}\cdot d\underline{a} \qquad \left[\begin{array}{c}\text{divergence}\\ \text{theorem}\end{array}\right]$$

$$\int_V \underline{A}\cdot(\nabla f)\,d\tau' = \int_S f\underline{A}\cdot d\underline{a} - \int_V f(\nabla\cdot\underline{A})\,d\tau' \qquad (7.5)$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

$\boxed{(5)\quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)}$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Return to $\nabla \cdot \underline{A}$

$$\frac{4\pi}{\mu_0}\left(\nabla \cdot \underline{A}\right) = \int_V \underline{j}(\underline{r}') \cdot \nabla\left(\frac{1}{R}\right) d\tau' = -\int_V \underline{j}(\underline{r}') \cdot \nabla_{r'}\left(\frac{1}{R}\right) d\tau'$$

(using 7.4)

Integrate by parts

$$= -\int_S \underline{j}(\underline{r}')\left(\frac{1}{R}\right) \cdot d\underline{a} + \int_V \underbrace{\left[\nabla_{r'} \cdot \underline{j}(\underline{r}')\right]}_{=0 \text{ in magnetostatics}} \frac{1}{R}\, d\tau'$$

by taking boundary to be very far from sources in a finite system $\Rightarrow 0$

$$\boxed{\therefore \quad \nabla \cdot \underline{A} = 0}$$

Suggestions for further reading on magnetostatics

Griffiths: "Introduction to Electrodynamics". Chapter 5.

Heald and Marion: "Classical Electromagnetic Radiation".
e.g., Sections 1.4, 1.5, 1.7, 2.6 & 2.7.

Rather more advanced level:

Jackson: "Classical Electrodynamics". Chapter 5.

In particular, for the (rather long) proof that $\nabla \times \underline{B} = \mu_0 \underline{j}$ starting from Biot-Savart see, e.g.,

Griffiths. Section 5.3.2.

Jackson. Section 5.3.