Lecture 7 Magnetostatics (continued)

In electrostatics consider $\underline{E}$ and V for a point charge q at $\underline{r}'$

$$\underbrace{\nabla \cdot \underline{E} = -\nabla^2 V}_{\Downarrow} = -\frac{1}{4\pi\varepsilon_0}\int_V \rho(\underline{r}')\,\nabla^2\left(\frac{1}{R}\right) d\tau'$$

are zero everywhere except at infinitesimally small point at location of charge

$$\nabla \cdot \left(\frac{\hat{\underline{R}}}{R^2}\right) = -\nabla^2\left(\frac{1}{R}\right) = 4\pi\delta^3(\underline{R}) = 4\pi\delta^3(\underline{r}-\underline{r}')$$

$$\therefore \quad \nabla \times \underline{B} = \nabla \times (\nabla \times \underline{A}) = -\nabla^2 \underline{A} + \cancel{\nabla(\nabla \cdot \underline{A})} = \frac{1}{4\pi\varepsilon_0}\int \underline{j}(\underline{r}')\left[-4\pi\delta(\underline{r}-\underline{r}')\right] d\tau' = \mu_0 \underline{j}$$

(in Coulomb gauge $\nabla \cdot \underline{A} = 0$)

Multipole Expansion in Magnetostatics

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\underline{l}'}_{\text{"monopole" term}=0} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\, d\underline{l}'}_{\text{"dipole" term}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left[\frac{3\cos^2\alpha - 1}{2}\right] d\underline{l}'}_{\text{"quadrupole" term}} + \ldots\ldots\right.$$

$$\underline{A}_{dipole}(r) = \frac{\mu_0}{4\pi}\,\frac{\underline{m}\times\hat{\underline{r}}}{r^2}$$

where $\underline{m} = I\,\underline{a}$ is the "magnetic dipole moment"

At the risk of belabouring this point let's include explicitly the "$(\underline{r})$"

$$\nabla \cdot \underline{E}(\underline{r}) = -\nabla^2 V(\underline{r}) = \frac{\rho(\underline{r})}{\varepsilon_0}$$

$$\nabla \times \underline{B}(\underline{r}) = -\nabla^2 \underline{A}(\underline{r}) = \mu_0 \underline{j}(\underline{r})$$

That is:

The $\nabla \cdot \underline{E}$ and $\nabla^2 V$ at a given point depend only on ρ at exactly that point!
" $\nabla \times \underline{B}$ " $\nabla^2 \underline{A}$ " " " " " " " $\underline{j}$ " " " "

The $\rho(\underline{r}')$ for all $\underline{r}' \neq \underline{r}$ have no effect on $\nabla \cdot \underline{E}$ and $\nabla^2 V$ at $\underline{r}$!
" $\underline{j}(\underline{r}')$ " " " " " " " $\nabla \times \underline{B}$ " $\nabla^2 \underline{A}$ " "

Summary of Lecture 7b)

Two useful mathematical techniques

1) $\nabla_{r'} f(\underline{r}-\underline{r}') = -\nabla f(\underline{r}-\underline{r}')$

symbol represents differentiation with respect to primed coordinates

2) Apply the idea of integration by parts to products in vector calculus

For example, using $\nabla\cdot(f\underline{A}) = f(\nabla\cdot\underline{A}) + \underline{A}\cdot(\nabla f)$ and the divergence theorem

$$\int_V \underline{A}\cdot(\nabla f)\, d\tau' = \int_S f\underline{A}\cdot d\underline{a} - \int_V f(\nabla\cdot\underline{A})\, d\tau'$$

boundary integral can often be zero

Can transfer action of ∇ from one element of product to the other.

This allows us to prove that our choice of

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{j}(\underline{r}')}{R}\, d\tau'$$

is consistent with $\nabla\cdot\underline{A} = 0$ as required by the proof in Lecture 7.