# Lecture 9) The Wave Equations for the Potentials and their Solutions

Start from inhomogeneous Maxwell Equations and substitute

$$\underline{E} = -\nabla V - \frac{\partial \underline{A}}{\partial t} \quad \text{and} \quad \underline{B} = \nabla \times \underline{A}$$

A) $$\nabla . \underline{E} = -\nabla^2 V - \frac{\partial}{\partial t}\left(\nabla . \underline{A}\right) = \rho / \varepsilon_0 \qquad (9.1)$$

B) $$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \left[ -\nabla\left(\frac{\partial V}{\partial t}\right) - \frac{\partial^2 \underline{A}}{\partial t^2} \right]$$

L.H.S. : $$\nabla\left(\nabla \times \underline{A}\right) = \nabla\left(\nabla . \underline{A}\right) - \nabla^2 \underline{A} \qquad [V.I.11]$$

~~R.H.S. :~~ $\cancel{\mu_0 \underline{j} + \mu_0 \varepsilon_0} \Big[$

$$\therefore \quad \mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla^2 \underline{A} + \nabla\left( \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} + \nabla . \underline{A} \right) = \mu_0 \underline{j} \qquad (9.2)$$

In Electrodynamics convenient to choose "Lorenz" gauge

$$\boxed{\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} + \nabla \cdot \underline{A} = 0} \qquad (9.3)$$

Insert (9.3) into (9.1) & (9.2)

$$\mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \rho/\varepsilon_0 \qquad (9.4)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla^2 \underline{A} = \mu_0 \underline{j} \qquad (9.5)$$

The inhomogeneous wave equations for $V$, $\underline{A}$

Since $\varepsilon_0 \mu_0 = \frac{1}{c^2}$ the potentials (fields) propagate at the speed of light

Define $\Box^2 \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ "D'Alembertian"

$$\left.\begin{aligned} \Box^2 V &= \rho/\varepsilon_0 \\ \Box^2 \underline{A} &= \mu_0 \underline{j} \end{aligned}\right\} \qquad (9.6)$$

# Notes

- Pleasing symmetry of equations for $V$ and $\underline{A}$
- Equation for $\underline{A}$ is 3 equations

$$\Box^2 A_i = \mu_0 j_i$$

- Contain essentially the same information as Maxwell's Equations but in a convenient 4-component form.

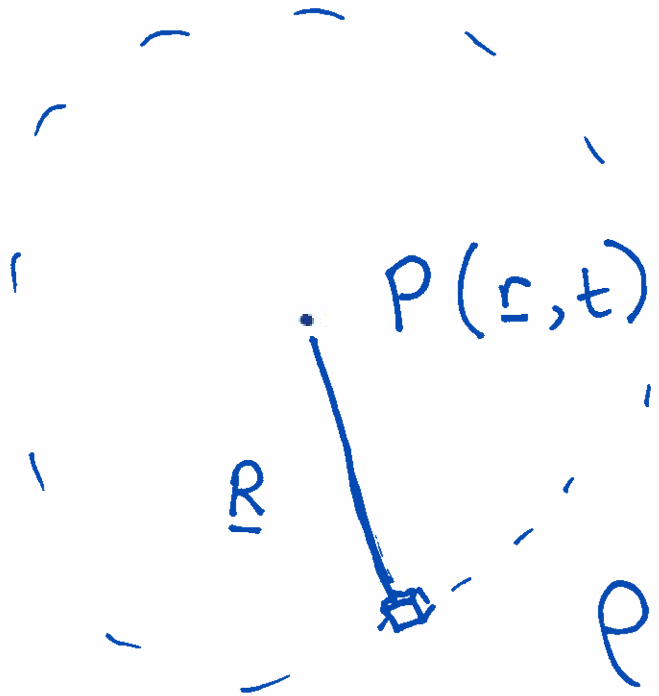
  (as opposed to 6 components of $\underline{E}$ & $\underline{B}$)

# Discuss the physical interpretation of wave equations

$$\frac{1}{c^2}\frac{\partial V}{\partial t^2} - \nabla^2 V = \rho/\varepsilon_0$$

Still have "Delta function" or "static" contribution

$$\left[\nabla^2 V\right]_{static} = -\frac{\rho(\underline{r}, t)}{\varepsilon_0} \qquad (9.7)$$




$$\rho(\underline{R}, t_{ret}) = \rho\left(\underline{R}, t - \frac{R}{c}\right)$$

Dynamic contribution

$$\left[\nabla^2 V\right]_{dynamic} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ from changing $\rho(\underline{R}, t_{ret})$

# Reminder of the potentials in electro-/magnetostatics

$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\underline{r}')}{R}\, d\tau' \quad \text{is a solution of} \quad \nabla^2 V = -\rho/\varepsilon_0$$

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\underline{j}(\underline{r}')}{R}\, d\tau' \quad \text{is a solution of} \quad \nabla^2 \underline{A} = -\mu_0 \underline{j}$$

[in the Coulomb gauge]

where $\underline{R} = \underline{r} - \underline{r}'$

## "Physics motivated" guess for the solutions $V$, $\underline{A}$ to the wave equations

Replace $\rho(\underline{r}')$ with $\rho(\underline{r}', t_{ret})$

$$V(\underline{r}, t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\underline{r}', t_{ret})}{R} \, d\tau' \qquad (9.8)$$

similarly

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\underline{j}(\underline{r}', t_{ret})}{R} \, d\tau \qquad (9.9)$$

The "retarded potentials"

We need to work out

$$\left[\nabla^2 V(\mathbf{r},t)\right] = \frac{1}{4\pi\varepsilon_0}\int \nabla^2 \left[\frac{\rho(\mathbf{r}', t_{ret})}{R}\right] d\tau' \quad (9.10)$$

We have to be careful: dependence on unprimed coordinates arise through $R$, but also $t_{ret} = t - \frac{R}{c}$ !

Look at Mini-Lecture 9b

can skip to ~9 minute into video.