

Central to our treatment of Special Relativity is the concept of a 4-vector

Archetypal example:

$$\underline{x} = (ct, \underline{r}) = (ct, x, y, z) = (ct, x^i) = (x^0, x^1, x^2, x^3) = x^\mu$$

$i = 1, 2, 3$ $(\mu = 0, 1, 2, 3)$

illustrates the main properties of all 4-vectors

- has the structure (scalar, vector)
- for convenience we arrange for all 4 components to have same units
- components transform according to the Lorentz Transformation

$$x'^0 = \gamma(x^0 - \beta x^1), \quad x'^1 = \gamma(-\beta x^0 + x^1), \quad x'^2 = x^2, \quad x'^3 = x^3$$

$\underline{x}(1) - \underline{x}(2) = \Delta \underline{x} = (c\Delta t, \Delta \underline{r})$ also transforms as a 4-vector

Exercise for you: verify this.

Scalar product of two 4-vectors

$$\underline{\underline{a}} \cdot \underline{\underline{b}} \equiv a^0 b^0 - \underline{a} \cdot \underline{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

has the same value in all inertial frames

$$\therefore \underline{\underline{a}}' \cdot \underline{\underline{b}}' = \underline{\underline{a}} \cdot \underline{\underline{b}}$$

or "is invariant with respect to a Lorentz transformation" or
"is Lorentz invariant" or "is a Lorentz scalar"

Exercise for you: verify this.

The "magnitude" of a 4-vector is just a special case

$$\underline{\underline{a}}' \cdot \underline{\underline{a}}' = \underline{\underline{a}} \cdot \underline{\underline{a}} \quad (\text{is invariant})$$

Example of using the scalar product

Consider the space-time interval between two events in S connected by a signal travelling at the speed of light.

$$\text{Thus: } \frac{\Delta r}{\Delta t} = c$$

$$\therefore (\Delta \tilde{x})^2 = (c\Delta t)^2 - (\Delta r)^2 = 0$$

$$(\Delta \tilde{x}') = 0 = (c\Delta t')^2 - (\Delta r')^2 = 0 \quad \therefore \frac{\Delta r'}{\Delta t'} = c$$

$\therefore$ Signal travels at speed of light in frame S'

Exercise for you: prove this result alternatively using the Lorentz transformation.

Additional exercises provided on the Special Relativity Revision sheet,