Central to our treatment of Special Relativity is the concept of a <u>4 - vector</u>

Archetypal example :

$$\underset{\sim}{x} = (ct, \underline{r}) = (ct, x, y, z) = (ct, x^i) = (x^0, x^1, x^2, x^3) = x^\mu$$

$i = 1, 2, 3$ $\quad$ $(\mu = 0, 1, 2, 3)$

illustrates the main properties of all 4-vectors

- has the structure (scalar, vector)
- for convenience we arrange for all 4 components to have same unit
- components transform according to the Lorentz Transformation

$$x'^0 = \gamma(x^0 - \beta x^1) , \quad x'^1 = \gamma(-\beta x^0 + x^1) , \quad x'^2 = x^2 , \; x'^3 = x^3$$

$\underset{\sim}{x}(1) - \underset{\sim}{x}(2) = \Delta \underset{\sim}{x} = (c\Delta t, \Delta \underline{r})$ also transforms as a 4 -vector

<u>Exercise for you</u> : verify this .

Q1) (b) $x'^0_{(i)} = \gamma\left(x^0_{(i)} - \beta x^1_{(i)}\right)$

$$x'^1_{(i)} = \gamma\left(-\beta x^0_{(i)} + x^1_{(i)}\right)$$

$$x'^2_{(i)} = x^2_{(i)}$$

$$x'^3_{(i)} = x^3_{(i)}$$

$$\therefore \quad \Delta x'^0 = x'^0_{(1)} - x'^0_{(2)} = \gamma\left(\left[x^0_{(1)} - x^0_{(2)}\right] - \beta\left[x^1_{(1)} - x^1_{(2)}\right]\right)$$

$$= \gamma\left(\Delta x^0 - \beta \Delta x^1\right)$$

$$\Delta x'^1 = x'^1_{(1)} - x'^1_{(2)} = \gamma\left(-\beta\left[x^0_{(1)} - x^0_{(2)}\right] + \left[x^1_{(1)} - x^1_{(2)}\right]\right)$$

$$= \gamma\left(-\beta \Delta x^0 + \Delta x^1\right)$$

$$\Delta x'^2 = x'^2_{(1)} - x'^2_{(2)} = x^2_{(1)} - x^2_{(2)} = \Delta x^2$$

and similarly for $\Delta x'^3 = \Delta x^3$

$\therefore$ Δx^μ obeys the Lorentz Transformations and is a 4-vector.

Scalar product of two 4-vectors

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \equiv a^0 b^0 - \underline{a} \cdot \underline{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

has the same value in all inertial frames

$$\therefore \quad \underset{\sim}{a}' \cdot \underset{\sim}{b}' = \underset{\sim}{a} \cdot \underset{\sim}{b}$$

or "is invariant with respect to a Lorentz transformation" or "is Lorentz invariant" or "is a Lorentz scalar"

Exercise for you: verify this.

The "magnitude" of a 4-vector is just a special case

$$\underset{\sim}{a}' \cdot \underset{\sim}{a}' = \underset{\sim}{a} \cdot \underset{\sim}{a} \qquad \text{(is invariant)}$$

Q2) $\underset{\sim}{a}' \cdot \underset{\sim}{b}' = a'^0 b'^0 - a'^1 b'^1 - a'^2 b'^2 - a'^3 b'^3$

$$= \gamma^2 \left(a^0 - \beta a^1\right)\left(b^0 - \beta b^1\right) - \gamma^2\left(-\beta a^0 + a^1\right)\left(-\beta b^0 + b^1\right) - a^2 b^2 - a^3 b^3$$

$$= \gamma^2\left(a^0 b^0 - a^1 b^1\right)\left(1 - \beta^2\right) - a^2 b^2 - a^3 b^3$$

$$= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

$= \underset{\sim}{a} \cdot \underset{\sim}{b}$, which is, therefore, an invariant.

Example of using the scalar product

Consider the space-time interval between two events in S connected by a signal travelling at the speed of light.

Thus : $\dfrac{\Delta r}{\Delta t} = c$

$\therefore \left(\Delta \underset{\sim}{x}\right)^2 = (c\Delta t)^2 - (\Delta r)^2 = 0$

$\left(\Delta \underset{\sim}{x}'\right) = 0 \quad = \left(c\Delta t'\right)^2 - \left(\Delta r'\right)^2 = 0 \quad \therefore \dfrac{\Delta r'}{\Delta t'} = c$

$\therefore$ Signal travels at speed of light in frame S'

Exercise for you: prove this result alternatively using the Lorentz transformation.

Additional exercises provided on the Special Relativity Revision sheet.

Q3) Signal travelling at light speed implies

$$\frac{(\Delta r)^2}{(c\Delta t)^2} = \frac{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}{(\Delta x^0)^2} = 1 \quad \text{(Eqn. A)}$$

In frame S'

$$(\Delta r')^2 = \gamma^2\left(\beta^2(\Delta x^0)^2 + (\Delta x^1)^2 - 2\beta\Delta x^0\Delta x^1\right) + (\Delta x^2)^2 + (\Delta x^3)^2$$

$$= \gamma^2\left(\beta^2\left[(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)\right] + (\Delta x^1)^2 - 2\beta\Delta x^0\Delta x^1\right) + (\Delta x^2)^2 + (\Delta x^3)^2$$

$\{$using (A)$\}$

$$= \gamma^2(1+\beta^2)(\Delta x^1)^2 + (1+\gamma^2\beta^2)\left[(\Delta x^2)^2 + (\Delta x^3)^2\right] - 2\beta\Delta x^0\Delta x^1\gamma^2$$

$$= \gamma^2\left[(1+\beta^2)(\Delta x^1)^2 + (\Delta x^2) + (\Delta x^3)^2 - 2\beta\Delta x^0\Delta x^1\right] \quad \text{(Eqn. B)}$$

$$\left\{\text{since } (1+\gamma^2\beta^2) = 1 + \frac{\beta^2}{1-\beta^2} = \gamma^2\right\}$$

$$(\Delta x'^0)^2 = \gamma^2\left((\Delta x^0)^2 + \beta^2(\Delta x^1)^2 - 2\beta\Delta x^0\Delta x^1\right)$$

$$= \gamma^2\left[(1+\beta^2)(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 - 2\beta\Delta x^0\Delta x^1\right]$$

(Eqn. C)

$\{$using (A)$\}$

Comparing (B) and (C) shows that $(\Delta r')^2/(\Delta x'^0)^2 = 1$

$\therefore$ Signal travels at speed c in S'.

NB Proof using invariant interval is much shorter!