Lecture 11) Special Relativity in the Minkowski Representation (index notation)

Motivated by the relative minus sign for the space vector part of the 4-vector scalar product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

$$a^\mu \equiv (a^0, a^1, a^2, a^3) \quad \text{a } \underline{\text{contravariant}} \text{ vector} \qquad (11.1)$$

$$a_\mu \equiv (a_0, a_1, a_2, a_3) \equiv (a^0, -a^1, -a^2, -a^3) \qquad (11.2)$$

a <u>covariant</u> vector

Eg. $x^\mu = (ct, x, y, z)$

$x_\mu = (ct, -x, -y, -z)$

The scalar product is written as

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a^\mu b_\mu = a_\mu b^\mu$$

Notes

1) Repeated indices: imply that repeated indices are summed over.

("The Einstein summation convention")

2) Only if there is one upper (contravariant) and one lower (covariant) index is the result a Lorentz invariant.

We can convert contravariant $\Longleftrightarrow$ covariant vectors using the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

$$g^{\mu\nu} a_\nu = a^\mu \quad \text{"raises" Lorentz index}$$

$$g_{\mu\nu} a^\nu = a_\mu \quad \text{"Lowers" Lorentz index.}$$

Verify this!

Exercises

1) Verify that $g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}{}_{\lambda} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times 4}$

2) If a^{μ} and b^{μ} are position (in space-time) 4-vectors then $a^{\mu} b^{\mu}$ is <u>not</u> a Lorentz invariant!

Minkowski representation of the Lorentz Transformations

Consider the contra-variant 4-vector $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

and $\underline{\beta} = \beta \hat{\underline{x}}$

$$x'^0 = \gamma(x^0 - \beta x^1) = \underbrace{\frac{\partial x'^0}{\partial x^0}}_{\gamma} x^0 + \underbrace{\frac{\partial x'^0}{\partial x^1}}_{-\gamma\beta} x^1 + \underbrace{\frac{\partial x'^0}{\partial x^2}}_{0} x^2 + \underbrace{\frac{\partial x'^0}{\partial x^3}}_{0} x^3$$

$$x'^1 = \gamma(-\beta x^0 + x^1)$$

$$x'^2 = x^2$$

$$x'^3 = x^3 \qquad (11.5)$$

<u>Writing in more general form</u>

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = \Lambda^\mu{}_\nu x^\nu = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad (11.6)$$

Exercise: Verify that (11.6) is equivalent to (11.5).

How does a covariant 4-vector transform?

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$$

$$x_0' = ct' = \gamma(ct - \beta x) = \gamma(x_0 + \beta x_1) = \frac{\partial x_0'}{\partial x_0} x_0 + \frac{\partial x_0'}{\partial x_1} x_1 + \ldots$$

$$x_1' = -x' = -\gamma(-\beta ct + x) = \gamma(\beta x_0 + x_1) = \frac{\partial x_1'}{\partial x_0} x_0 + \frac{\partial x_1'}{\partial x_1} x_1 + \ldots$$

(11.7)

or $$x'_\mu = \frac{\partial x'_\mu}{\partial x_\nu} x_\nu = (\Lambda^{-1})^\nu{}_\mu \, x_\nu = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(11.8)

Exercises

1) Verify that $\Lambda^{\mu}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\lambda} = \delta^{\mu}{}_{\lambda} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times 4}$

2) $$x^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} x'^{\nu} = (\Lambda^{-1})^{\mu}{}_{\nu} x'^{\nu} \qquad (11.9)$$

3) $$x_{\mu} = \frac{\partial x_{\mu}}{\partial x'_{\nu}} x'_{\nu} = \Lambda^{\nu}{}_{\mu} x'_{\nu} \qquad (11.10)$$

Let's define a 4-dimensional "grad" operator and work out how this transforms

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} = \left(\Lambda^{-1}\right)^{\nu}{}_{\mu} \frac{\partial}{\partial x^{\nu}}$$

from (11.9)

$$\frac{\partial}{\partial x'_{\mu}} = \frac{\partial x_{\nu}}{\partial x'_{\mu}} \frac{\partial}{\partial x_{\nu}} = \Lambda^{\mu}{}_{\nu} \frac{\partial}{\partial x_{\nu}}$$

from (11.10)

That is: differential w.r.t. $\left\{\begin{array}{ll}\text{contravariant} & x^{\mu} \\ \text{covariant} & x_{\nu}\end{array}\right\}$ transforms as $\left\{\begin{array}{l}\text{covariant} \\ \text{contravariant}\end{array}\right\}$

Let's define the following shorthand:

$$\boxed{\begin{aligned} \partial_\mu &= \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \\ \partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \end{aligned}}$$