Notes

1) Repeated indices: imply that repeated indices are summed over.

("The Einstein summation convention")

2) Only if there is one upper (contravariant) and one lower (covariant) index is the result a Lorentz invariant.

We can convert contravariant $\Longleftrightarrow$ covariant vectors using the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

$$\left.\begin{array}{lll} g^{\mu\nu} a_\nu = a^\mu & \text{"raises" Lorentz index} \\ g_{\mu\nu} a^\nu = a_\mu & \text{"Lowers" Lorentz index.} \end{array}\right\} \text{Verify this!}$$

Exercises

1) Verify that $g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}{}_{\lambda} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times 4}$

2) If a^{μ} and b^{μ} are position (in space-time) 4-vectors then $a^{\mu} b^{\mu}$ is <u>not</u> a Lorentz invariant!

Minkowski representation of the Lorentz Transformations

Consider the contra-variant 4-vector $x^{\mu} = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

and $\underline{\beta} = \beta \hat{\underline{x}}$

$$x'^0 = \gamma(x^0 - \beta x^1) = \underbrace{\frac{\partial x'^0}{\partial x^0}}_{\gamma} x^0 + \underbrace{\frac{\partial x'^0}{\partial x^1}}_{-\gamma\beta} x^1 + \underbrace{\frac{\partial x'^0}{\partial x^2}}_{0} x^2 + \underbrace{\frac{\partial x'^0}{\partial x^3}}_{0} x^3$$

$$x'^1 = \gamma(-\beta x^1 + x^1)$$

$$x'^2 = x^2$$

$$x'^3 = x^3 \qquad (11.5)$$

Writing in more general form

$$x'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} x^{\nu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad (11.6)$$

Exercise: Verify that (11.6) is equivalent to (11.5).

Exercises

1) Verify that $\Lambda^{\mu}{}_{\nu} (\Lambda^{-1})^{\nu}{}_{\lambda} = \delta^{\mu}{}_{\lambda} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \mathbb{1}_{4\times4}$

2) $x^{\mu} = \dfrac{\partial x^{\mu}}{\partial x'^{\nu}} x'^{\nu} = (\Lambda^{-1})^{\mu}{}_{\nu} x'^{\nu}$ (11.9)

3) $x_{\mu} = \dfrac{\partial x_{\mu}}{\partial x'_{\nu}} x'_{\nu} = \Lambda^{\nu}{}_{\mu} x'_{\nu}$ (11.10)