Writing out the Lecture 11 exercises as matrices

$$g^{\mu\nu} a_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = a^\mu$$

$$g_{\mu\nu} a^\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} a^0 \\ -a^1 \\ -a^2 \\ -a^3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_\mu$$

$$g^{\mu\nu} g_{\nu\lambda} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$a^\mu b^\mu = a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$= \gamma\left(a'^0 + \beta a'^1\right)\gamma\left(b'^0 + \beta b'^1\right) + \gamma\left(a'^1 + \beta a'^0\right)\gamma\left(b'^1 + \beta b'^0\right) + a'^2 b'^2 + a'^3 b'^3$$

$$= \gamma^2\left[\left(1+\beta^2\right)\left(a'^0 b'^0 + a'^1 b'^1\right) + 2\beta\left(a'^0 b'^1 + a'^1 b'^0\right) + a'^2 b'^2 + a'^3 b'^3\right.$$

$$\neq a'^\mu b'^\mu$$

$\therefore$ $a^\mu b^\mu$ is not a Lorentz scalar

Eqn (11.6) gives
$$x'^\mu = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \gamma x^0 - \gamma\beta x^1 \\ -\gamma\beta x^0 + \gamma x^1 \\ x^2 \\ x^3 \end{pmatrix},$$

which is equivalent to the four equations of (11.5)

$$\Lambda^{\mu}{}_{\nu}\left(\Lambda^{-1}\right)^{\nu}{}_{\lambda} = \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & \multicolumn{2}{c}{0} \\ -\gamma\beta & \gamma & & \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} \gamma & \gamma\beta & \multicolumn{2}{c}{0} \\ \gamma\beta & \gamma & & \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} \gamma^2(1-\beta^2) & \gamma^2(\beta-\beta) & & \\ \gamma^2(-\beta+\beta) & \gamma^2(-\beta^2+1) & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & 1 & & \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right)$$

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, x'^{\nu} = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & \multicolumn{2}{c}{0} \\ \gamma\beta & \gamma & & \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right) \begin{pmatrix} x'^{0} \\ x'^{1} \\ x'^{2} \\ x'^{3} \end{pmatrix} = \begin{pmatrix} \gamma x'^{0} + \gamma\beta x'^{1} \\ \gamma\beta x'^{0} + \gamma x'^{1} \\ x'^{2} \\ x'^{3} \end{pmatrix} = \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

$$= x^{\mu}$$, as required by Eqn. (11.9)

Using index notation $x_\mu = \Lambda^\nu{}_\mu x'_\nu$ may be written as

$$x_0 = \Lambda^\nu{}_0 x'_\nu = \Lambda^0{}_0 x'_0 + \Lambda^1{}_0 x'_1 \quad (\text{since } \Lambda^2{}_0 = \Lambda^3{}_0 = 0)$$

$$= \gamma x'_0 - \gamma\beta x'_1$$

$$x_1 = \Lambda^\nu{}_1 x'_\nu = \Lambda^0{}_1 x'_0 + \Lambda^1{}_1 x'_1 = -\gamma\beta x'_0 + \gamma x'_1$$

$$x_2 = \Lambda^\nu{}_2 x'_\nu = \Lambda^2{}_2 x'_2 = x'_2, \text{ and similarly } x_3 = x'_3 .$$

Alternatively, writing x'_ν as a row vector*

$$\begin{pmatrix} x'_0 & x'_1 & x'_2 & x'_3 \end{pmatrix} \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & \\ -\gamma\beta & \gamma & & \\ \hline & & 1 & 0 \\ 0 & & 0 & 1 \end{array}\right) = \begin{pmatrix} \gamma x'_0 - \gamma\beta x'_1, & -\gamma\beta x'_0 + \gamma x'_1 & x'_2 & x'_3 \end{pmatrix}$$

Both methods give the correct transformation equations for x_μ in terms of x'_μ. (cf equations 11·7 for x'_μ in terms of x_μ)

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* See note on writing 4-vectors as matrices