Lecture 12a) Proper time and building 4-vectors for Electrodynamics

Summary of Special Relativity in Minkowski Notation (Lecture 11)

Contravariant (upper index)

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

Lorentz Transformation

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(-\beta x^0 + x^1)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\Lambda^\mu{}_\nu = \left[\begin{array}{cc|c} \gamma & -\gamma\beta & \\ -\gamma\beta & \gamma & 0 \\ \hline & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \multicolumn{2}{c|}{0} & \end{array}\right]$$

Covariant (lower index)

$$x_\mu = (x_1, x_2, x_3, x_4) = (ct, -x, -y, -z)$$

$$x'_0 = \gamma(x_0 + \beta x_1)$$
$$x'_1 = \gamma(\beta x_0 + x_1)$$
$$x'_2 = x_2$$
$$x'_3 = x_3$$

$$x'_\mu = (\Lambda^{-1})^\nu{}_\mu x_\nu$$

$$(\Lambda^{-1})^\nu{}_\mu = \left[\begin{array}{cc|c} \gamma & \gamma\beta & \\ \gamma\beta & \gamma & 0 \\ \hline & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \multicolumn{2}{c|}{0} & \end{array}\right]$$

By considering the Lorentz transformations in both directions $x \rightarrow x'$ and $x' \rightarrow x$, for contra- and co-variant $\underset{\sim}{x}$ we can show that:

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\nu} = \frac{\partial x'_{\nu}}{\partial x_{\mu}} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}$$

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{\partial x_{\nu}}{\partial x'_{\mu}}$$

Metric Tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} \qquad g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu{}^\nu = \mathbb{1}_{4\times4}$$

$x^\mu = g^{\mu\nu} x_\nu$ "raises Lorentz index"

$x_\mu = g_{\mu\nu} x^\nu$ "lowers Lorentz index"

Scalar Product

$$a_\mu b^\mu = a^\mu b_\mu = a_\mu g^{\mu\nu} b_\nu = a^\mu g_{\mu\nu} b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

is a Lorentz scalar (invariant) $a'_\mu b'^\mu = a_\mu b^\mu$

Magnitude

$x_\mu x^\mu = (ct)^2 - x^2 - y^2 - z^2$ is also Lorentz invariant.

Summary of Lecture 11 The operator ∂^μ

We define :

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \therefore \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \left(\Lambda^{-1}\right)^\nu{}_\mu \frac{\partial}{\partial x^\nu}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) \therefore \partial'^\mu = \frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu}$$

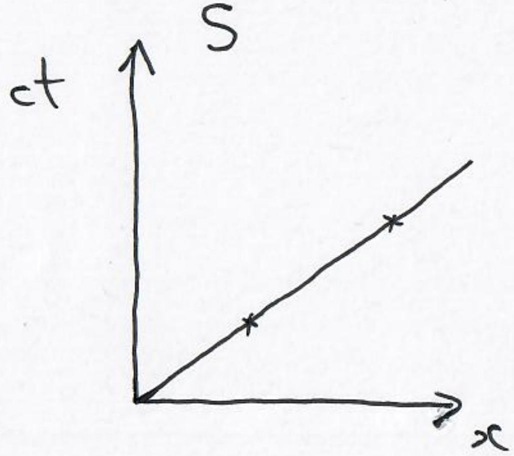
That is :

Differential w.r.t. contravariant, x^μ, transforms as covariant 4-vector

" " covariant, x_ν, " " contravariant "

Lecture 12a) Proper time and building 4-vectors for Electrodynamics

Proper Time



Consider the space-time interval between two space-time points on the world line of a particle

In the rest frame of the particle, S', the events occur at the same point in space

$$\Delta x'^{i} = 0$$

$$\Delta x'^{\mu} \Delta x'_{\mu} = c^2 \Delta t'^2 - 0 \equiv c^2 \Delta \tau^2$$

proper time interval

$\therefore$ Proper time is a Lorentz scalar

N.B. Proper time intervals can be measured by a single clock!

In frame S, in which the particle is moving with velocity $\beta c \hat{\underline{x}}$

L.T. $\Delta x^0 = c\Delta t = \gamma(c\Delta\tau + 0)$ or $\Delta t = \gamma \Delta \tau$

Time interval is always $\geqslant \Delta\tau$

Let's use proper time τ to extend the concept of velocity

$$\underline{u} = \frac{d\underline{r}}{dt}$$

of our moving particle.

Let's try to find a "4-velocity" (which is a 4-vector)

$$u^\mu = \frac{dx^\mu}{d\tau}$$

$dx^\mu \leftarrow$ a 4-vector

$d\tau \leftarrow$ is a Lorentz invariant

$\therefore$ u^μ is a 4-vector!

"Hybrid" object: dx^μ measured in frame S in which particle is moving

$d\tau$ measured in frame S' in which particle is stationary

Components

For $i = 1,2,3$: $\quad u^i = \dfrac{dx^i}{d\tau} = \gamma \dfrac{dx^i}{dt}$ i.e. γ times the "normal" velocity.

$$u^0 = \frac{dx^0}{d\tau} = \gamma c \frac{dt}{dt} = \gamma c$$

(since $\gamma \, d\tau = dt$)

$$\Rightarrow \quad u^{\mu} = (\gamma c, \gamma \underline{u})$$

Lorentz Transformation $\qquad u'^{\mu} = \Lambda^{\mu}{}_{\nu} u^{\nu}$

Magnitude $\qquad u_{\mu} u^{\mu} = \gamma^2 c^2 - \gamma^2 u^2 = c^2$

In the rest frame $\gamma = 1$, $\underline{u} = 0$
(magnitude is Lorentz invariant)

The 4-velocity is not so useful in of itself. However, it forms the basis for two very important 4 vectors $\Rightarrow$ Lecture 12b.