Lecture 12b) 4-vectors for Electrodynamics (part 2)

1) 4-momentum of a particle

Define $p^\mu = m u^\mu = m \frac{dx^\mu}{d\tau} = \gamma m (c, \underline{u})$

"mass of particle in its rest frame"

"rest mass" or "invariant mass" or just simply "mass"

p^μ <u>must</u> be a 4-vector since u^μ is a 4-vector and m is an invariant.

Reminder from Lecture 12a

Let's try to find a "4-velocity" (which is a 4-vector)

$$u^\mu = \frac{dx^\mu}{d\tau}$$

$dx^\mu \leftarrow$ a 4-vector

$d\tau \leftarrow$ is a Lorentz invariant

$\therefore$ u^μ is a 4-vector!

"Hybrid" object: dx^μ measured in frame S in which particle is moving

$d\tau$ measured in frame S' in which particle is stationary

Components

For $i = 1, 2, 3$: $\boxed{u^i} = \dfrac{dx^i}{d\tau} = \boxed{\gamma \dfrac{dx^i}{dt} \quad \text{i.e. } \gamma \text{ times the "normal" velocity}}$

$$\boxed{u^0} = \frac{dx^0}{d\tau} = \gamma c \frac{dt}{dt} = \boxed{\gamma c}$$

(since $\gamma\, d\tau = dt$)

Components of p^μ

For $i = 1, 2, 3$: $p^i = \gamma m u^i$

The component of the (relativistic) momentum

$$p^0 = \gamma m c = \frac{\gamma m c^2}{c} = \frac{E}{c},$$

where $E = \gamma m c^2$ is the total (relativistic) energy.

$$\Rightarrow \quad p^\mu = \left(\frac{E}{c}, \underline{p} \right)$$

Lorentz Transformation

$$p'^\mu = \Lambda^\mu{}_\nu \, p^\nu$$

Magnitude

$$\left.\begin{array}{l}\text{In rest frame } \gamma = 1 , \ \underline{u} = 0 \\ \qquad u_\mu u^\mu = c^2 \end{array}\right\} \quad p^\mu p_\mu = (mc)^2$$

Since this must be true in all frames

$$\left(\frac{E}{c}\right)^2 - p^2 = (mc)^2$$

$$E^2 - (pc)^2 = (mc^2)^2$$

2) 4 - current

Define $j^\mu = \rho_0 u^\mu$

$\uparrow$ the charge density in the rest frame of the charge distribution.

(which is Lorent invariant by definition)

$\therefore$ j^μ <u>must</u> be a 4-vector since: u^μ is a 4-vector

ρ_0 is an invariant.

Components of j^μ

For $i = 1, 2, 3$ $\quad j^i = \underbrace{\gamma \rho_0}_{} u^i$

charge density in frame S, in which charge distribution is moving.

In the rest frame S' we can write

$$\rho_0 = \frac{Q}{[\Delta x' \Delta y' \Delta z']}$$

Q ← total charge in volume element

$[\Delta x' \Delta y' \Delta z']$ ← volume

In frame S, in which the charge distribution is moving we have to measure simutaneously ($\Delta t = 0$) the two ends of Δx

L.T. $\quad \Delta x' = \gamma(\Delta x - 0) \quad \therefore \quad \Delta x = \frac{\Delta x'}{\gamma}$ "length contractio

$\therefore$ In frame S charge density $\rho = \dfrac{Q}{\Delta x \Delta y \Delta z} = \gamma \rho_0$

N.B. This assumes that Q is a Lorentz invariant, which turns out to be a good assumption.

Also $j^0 = \gamma \rho_0 c$

$$\Rightarrow j^\mu = (\gamma \rho_0 c, \gamma \rho_0 \underline{u}) = (\rho c, \rho \underline{u}) = (\rho c, \underline{j})$$

L.T. For you to fill out!

<u>Magnitude</u> $j^\mu j_\mu = (\rho_0 c)^2$

An application of this new notation:

The continuity equation for charge

$$\frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \nabla \cdot \underline{j} = 0$$

can be written as

$$\frac{\partial j^{\mu}}{\partial x^{\mu}} = \boxed{\partial_{\mu} j^{\mu} = 0}$$

Product of one contravariant and one covariant 4-vector.
This guarantees that also $\partial'_{\mu} j'^{\mu} = 0$

"Lorentz covariance": Laws of physics take the same form in all frames
(as required by 1st postulate of Special Relativity