Lecture 13) The 4-Potential and the Wave Equation in Lorentz-covariant form

We can write the D'Alembertian as

$$\partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box^2$$

Product of covariant and contravariant 4-vectors guarantees that $\Box^2$ is a Lorentz invariant

We can write the wave equations for the potentials as:

$$\left.\begin{aligned} \Box^2\left(\frac{V}{c}\right) &= \frac{\rho}{c\varepsilon_0} = \mu_0(\rho c) \\ \Box^2 \underline{A} &= \mu_0 \underline{j} \end{aligned}\right\}$$

R.H.S. is a constant multplying the components of j^μ

L.H.S. is a Lorentz invariant multiplying components of a new 4-vecto

Define $A^\mu \equiv \left(\frac{V}{c}, \underline{A}\right)$ allows us to write the wave equation in a very compact form

$$\boxed{\Box^2 A^\mu = \mu_0 j^\mu}$$

As already noted the properties of $\Box^2$, μ_0, j^μ under a Lorentz transformation guarantee that A^μ is a 4-vector.

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

The Lorenz Gauge Condition

$$\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{V}{c}\right] + \nabla \cdot \underline{A} = 0$$

can be written as

$$\boxed{\partial_\mu A^\mu = 0}$$

Written in this way the Lorenz gauge condition:

- seems like a natural extension of the Coulomb gauge condition $\nabla \cdot \underline{A} = 0$ to 4-dimensions
- is manifestly Lorentz covariant (which is not the case for $\nabla \cdot A = 0$).

General Gauge Transformation

$$\frac{V}{c} \Rightarrow \frac{V}{c} - \frac{1}{c}\frac{\partial \psi}{\partial t}$$

$$\underline{A} \Rightarrow \underline{A} + \nabla \psi$$

can be written as:

$$A^\mu \Rightarrow A^\mu - \partial^\mu \psi \qquad \left(\text{since } \partial^\mu = \frac{1}{c}\frac{\partial}{\partial t}, -\nabla \right)$$

Evaluation of $\underline{E}$ and $\underline{B}$ fields from A^μ

$$\frac{\underline{E}}{c} = -\nabla\left(\frac{V}{c}\right) - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}$$

Look at components $i = 1, 2, 3$ and write R.H.S. in terms of A^μ and ∂^μ

$$\frac{E_i}{c} = -\frac{\partial A^0}{\partial x^i} - \frac{\partial A^i}{\partial x^0} = \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0} = \partial^i A^0 - \partial^0 A^i$$

$$= F^{i0} = -F^{0i}$$

N.B. E_i, B_i are <u>not</u> components of 4-vectors.

They do <u>not</u> have contravariant and covariant forms, as far as we are concerned in this course!

Also,

$$B_i = (\nabla \times \underline{A})_i = \varepsilon_{ijk} \frac{\partial A^k}{\partial x^j} = -\varepsilon_{ijk} \frac{\partial A^k}{\partial x_j} = -\varepsilon_{ijk} \partial^j A^k$$

$$\equiv -F^{jk} = F^{kj}$$

e.g.,

$$B_1 = -\partial^2 A^3 + \partial^3 A^2 = -F^{23} = F^{32}$$

Putting all this together we have

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{array}{cc} & \longrightarrow \nu \\ \begin{array}{c}\downarrow \\ \mu\end{array} & \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{bmatrix} \end{array}$$

The "electromagnetic field tensor"

Exercise : Verify explicitly each element of $F^{\mu\nu}$