Lecture 15a) $\underline{E}$ and $\underline{B}$ fields produced by a moving point charge

Method 1 From A^μ in frame S, in which the charge is moving.

For reference, in S' we can write

$$\frac{\underline{E}'}{c} = \frac{q}{4\pi\varepsilon_0 c} \frac{\underline{R}'}{(R')^3}$$

$$\frac{E_1'}{c} = \frac{q}{4\pi\varepsilon_0 c} \frac{x'^1}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c} \frac{\gamma\left[x^1 - \beta x^0\right]}{(R')^3} \qquad (15.1)$$

$$\left.\begin{aligned} \frac{E_2'}{c} &= \frac{q}{4\pi\varepsilon_0 c} \frac{x'^2}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c} \frac{x^2}{(R')^3} \\ \frac{E_3'}{c} &= \frac{q}{4\pi\varepsilon_0 c} \frac{x^3}{(R')^3} \end{aligned}\right\} \qquad (15.2)$$

Potentials in frame S (From Lecture 14)

L.T. $\frac{V}{c} = A^0 = \gamma\left(A'^0 + \beta A'^1\right) = \gamma A'^0$

(Note: inverse Lorentz Transformation.)

$$\therefore\ A^0 = \frac{q}{4\pi\varepsilon_0 c}\,\gamma\,\frac{1}{\left[\left(\gamma\left[x^1-\beta x^0\right]\right)^2+\left(x^2\right)^2+\left(x^3\right)^2\right]^{1/2}}$$

$$A^1 = \gamma\left(\beta A'^0 + A'^1\right) = \gamma\beta A'^0 = \beta A^0$$

$$= \frac{q}{4\pi\varepsilon_0 c}\,\gamma\beta\,\frac{1}{\left[\left(\gamma\left[x^1-\beta x^0\right]\right)^2+\left(x^2\right)^2+\left(x^3\right)^2\right]^{1/2}}$$

$$A^2 = A^3 = 0$$

Exercise for you: Cross-check that $A'^\mu A'_\mu = A^\mu A_\mu$.

From the expressions for A^μ in frame S (Lecture 14) and rules for finding $\underline{E}$, $\underline{B}$ from A^μ (Lecture 13)

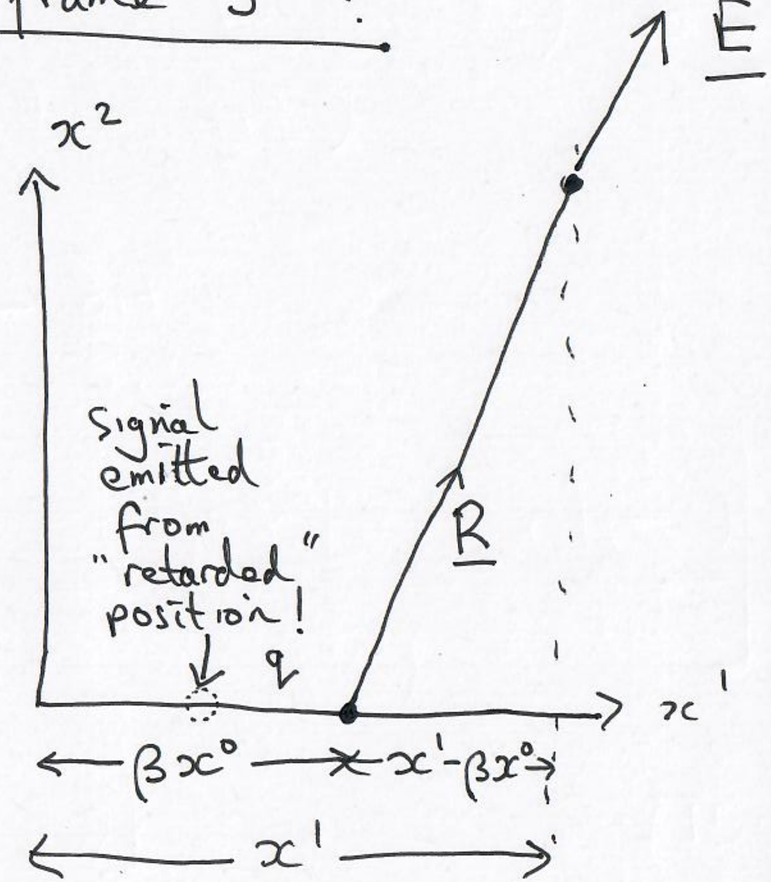
$$\frac{E_1}{c} = F^{10} = \partial^1 A^0 - \partial^0 A^1 = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0}$$

$$= -\frac{q\gamma}{4\pi\varepsilon_0 c}\left[\frac{\left(-\frac{1}{2}\right)2\gamma^2\left[x^1-\beta x^0\right]}{(R')^3} + \frac{\beta\left(-\frac{1}{2}\right)2\gamma^2\left[x^1-\beta x^0\right](-\beta)}{(R')^3}\right]$$

$$= \frac{q\gamma}{4\pi\varepsilon_0 c}\frac{\left[x^1-\beta x^0\right]}{(R')^3}\underbrace{\left[\gamma^2\left(1-\beta^2\right)\right]}_{1} \qquad (15.3)$$

$$\left.\begin{aligned} \frac{E_2}{c} &= F^{20} = \partial^2 A^0 - \partial^0 A^2 = -\frac{\partial A^0}{\partial x^2} = \frac{q\gamma}{4\pi\varepsilon_0 c}\frac{x^2}{(R')^3} \\ \frac{E_3}{c} &= \qquad\qquad\qquad\qquad\quad = \frac{q\gamma}{4\pi\varepsilon_0 c}\frac{x^3}{(R')^3} \end{aligned}\right\} \qquad (15.4)$$

What do these fields "look like" in frame S?



We can write

$$\frac{\underline{E}}{c} = \frac{q\gamma}{4\pi\varepsilon_0 c}\,\frac{\underline{R}}{(R')^3}$$

where

$$\underline{R} = \left[x^1 - \beta x^0\right]\underline{\hat{x}}^1 + x^2\underline{\hat{x}}^2 + x^3\underline{\hat{x}}^3$$

is the vector from the <u>current</u> position of q to the point $P(x^0, x^1, x^2, x^3)$ at which $\underline{E}$ is evaluated.

$\therefore$ $\underline{E}$ at time t is radial with respect to the position of q at time t

... and not at retarded time t_{ret} ! ?

Compare $\underline{E}$ and $\underline{E}'$ at fixed R'

From (15.1) and (15.3)

$$E_1 = E_1' \qquad (15.5)$$

From (15.2) and (15.4)

$$\begin{aligned} E_2 &= \gamma E_2' \\ E_3 &= \gamma E_3' \end{aligned} \qquad (15.6)$$

Notes:

1) Transverse components of $\underline{E}$ gain factor γ in S.
2) In addition, x' component of R' is Lorentz-contracted in S.

Sphere in S' is compressed along x' in S!

Magnitude of $\underline{E}$ in S

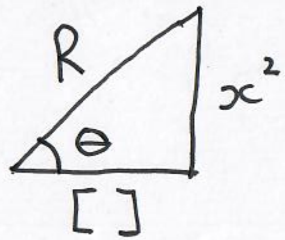
Given symmetry around $\hat{\underline{x}}^1$ let's consider $\underline{E}$ in the $x^1 - x^2$ plane

$$|\underline{E}| = \left(E_1^2 + E_2\right)^{1/2}$$

$$= \frac{q\gamma}{4\pi\varepsilon_0}\left(\frac{[\;]^2 + (x^2)^2}{\{\gamma^2[\;]^2 + (x^2)^2\}^3}\right)^{1/2}$$

$[\;] = [x^1 - \beta x^0]$

uses (15.3) (15.4)



$$R^2 = [\;]^2 + (x^2)^2$$

$$\sin\theta = x^2/R \,, \qquad \cos\theta = [\;]/R$$

$$|E| = \frac{q\gamma}{4\pi\varepsilon_0}\frac{1}{R^2}\left(\frac{1}{\gamma^2(1-\sin^2\theta) + \sin^2\theta}\right)^{3/2}$$

$\because \frac{1}{\gamma^2} = 1 - \beta^2$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\frac{1}{\gamma^2}\left(\frac{1}{1-\beta^2\sin^2\theta}\right)^{3/2} \qquad (15.7)$$

Exercise: verify these steps.

Now considering $|E|$ at fixed R

Along the direction of motion $(\sin\theta = 0)$

$|\underline{E}|$ decreased by factor $\frac{1}{\gamma^2}$ relative to stationary charge in S

Transverse $(\sin\theta = 1)$

$|\underline{E}|$ increased by factor γ relative to stationary charge in S

Electric Field Produced by a Point Charge Moving with Constant Velocity

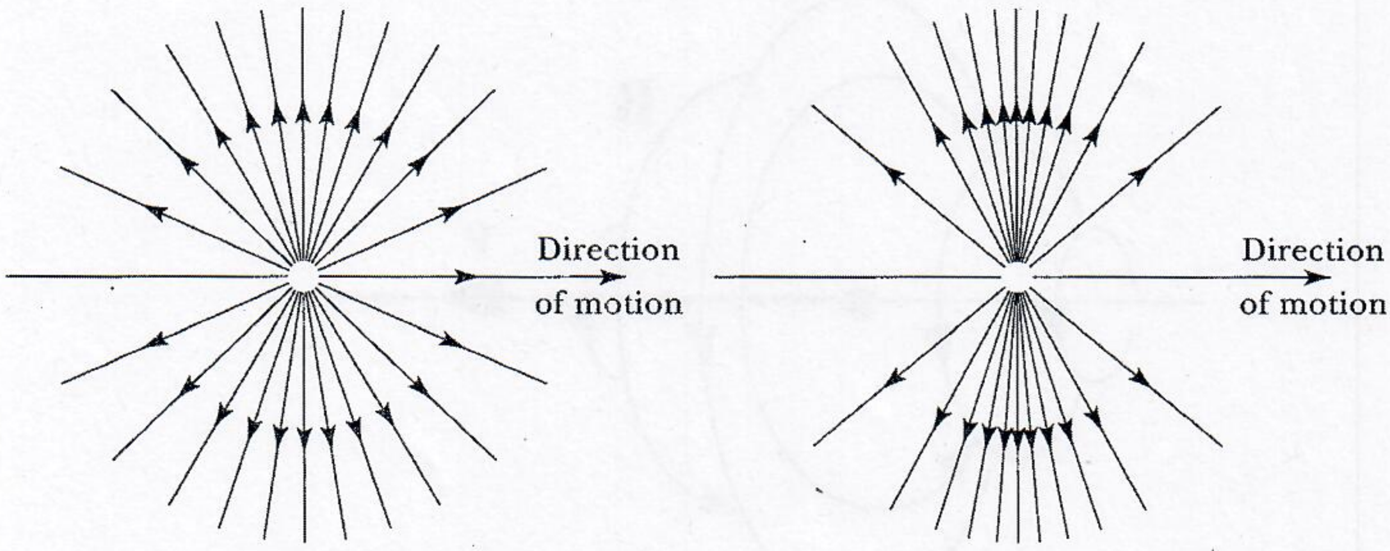


Figure 1: Electric field lines for $\beta = 0.7$ (left) and $\beta = 0.95$ (right) [1].

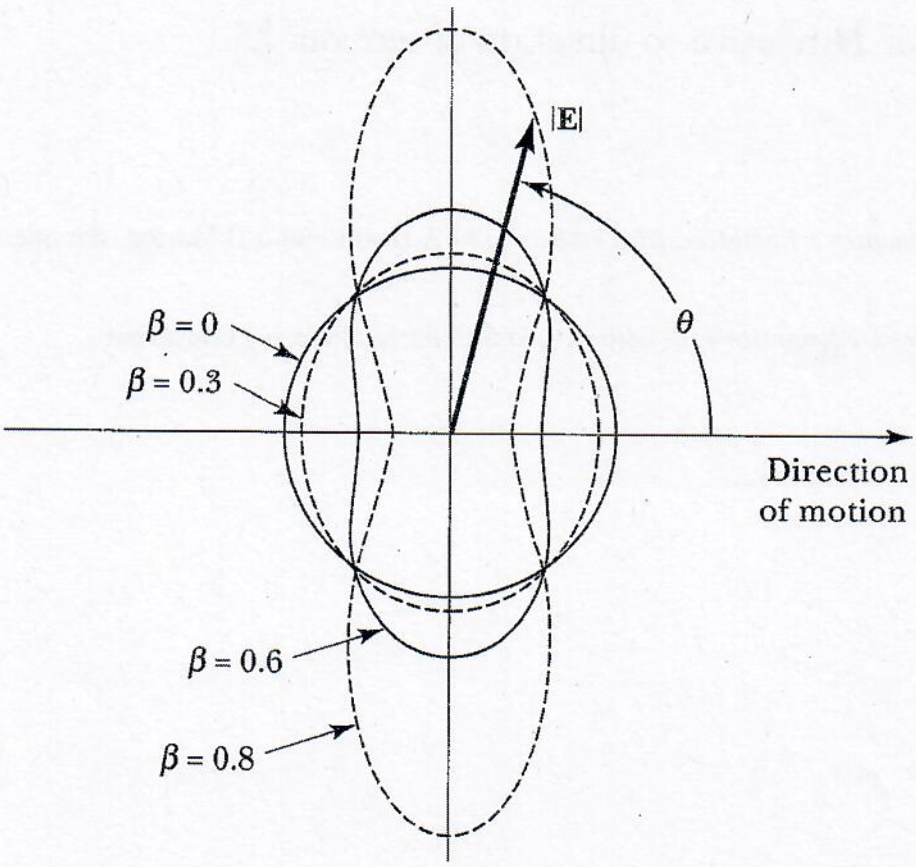


Figure 2: $|\boldsymbol{E}|$ at constant distance, R, from a moving point charge, shown as function of angle to direction of motion, θ, for different values of β [1].

B field for a moving point charge

$$B_i = \left[\nabla \times \underline{A}\right]_i = \varepsilon_{ijk} \frac{\partial A^k}{\partial x^j} = -\varepsilon_{ijk} \partial^j A^k = -F^{jk}$$

where $A^1 = \dfrac{q}{4\pi\varepsilon_0 c} \dfrac{\gamma\beta}{R'}$, $A^2 = A^2 = 0$

$$\left.\begin{aligned}
B_1 &= 0 \\
B_2 &= \frac{\partial A^1}{\partial x^3} = \frac{\left(-\frac{1}{2}\right) 2x^3}{(R')^3} \frac{q\gamma\beta}{4\pi\varepsilon_0 c} = -\frac{q\gamma\beta}{4\pi\varepsilon_0 c} \frac{x^3}{(R')^3} = -\frac{\beta E_3}{c} \\
B_3 &= -\frac{\partial A^1}{\partial x^2} = \frac{q\gamma\beta}{4\pi\varepsilon_0 c} \frac{x^2}{(R')^3} = \frac{\beta E_2}{c}
\end{aligned}\right\} \quad (15.8)$$

$\therefore$ we can write

$$\underline{B} = \frac{1}{c} \underline{\beta} \times \underline{E} \qquad (15.9)$$

Magnetic Field Produced by a Point Charge Moving with Constant Velocity

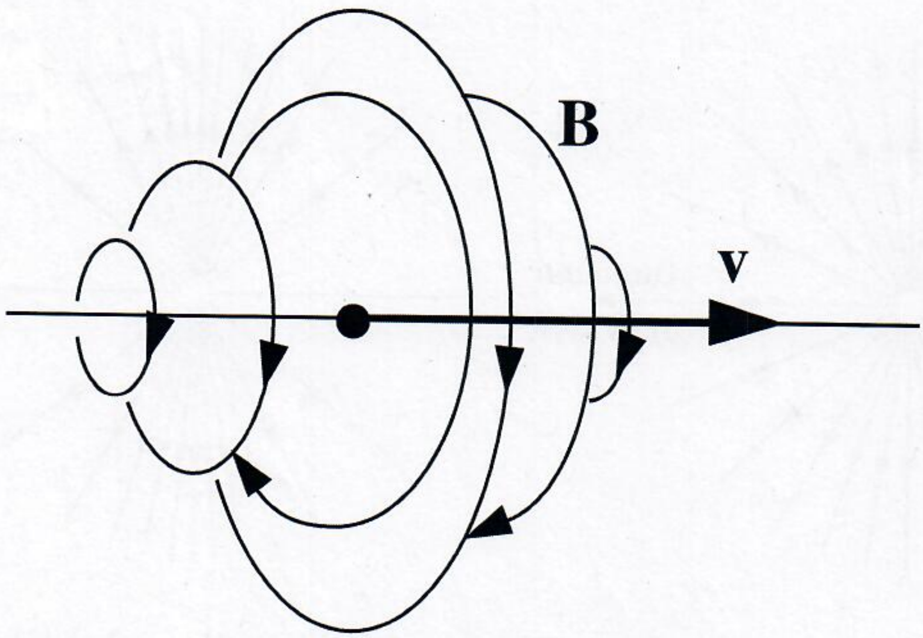


Figure 3: Lines of $\boldsymbol{B}$ relative to direction of motion. [2].

References

[1] *'Classical Electromagnetic Radiation (3rd edition)'*, M.A.Heald and J.B.Marion, Saunders College Publishing.

[2] *'Introduction to Electrodynamics (4th edition)'*, D.J.Griffiths, Pearson Education.