Lecture 15b) Alternative calculation of the fields produced by a point charge moving with a constant velocity

- Start with the E and B fields in the rest frame S'
- Work out how to transform $F^{\mu\nu}$.

Since $F^{\mu\nu}$ is a "tensor of rank 2"

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

Looks horendous perhaps, but index notation makes these calculations relatively straightforward!

Reminder

$$\Lambda^{\delta}{}_{\gamma} = \begin{array}{cc} & \gamma \longrightarrow \\ & \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \delta \downarrow \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

$$F^{\alpha\beta} = \begin{array}{cc} & \beta \longrightarrow \\ & \begin{array}{cccc} 0 & \;\;\;1 & \;\;\;2 & \;\;\;3 \end{array} \\ \alpha \downarrow \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left(\begin{array}{cccc} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{array}\right) \end{array}$$

Let's look at some specific examples.

$$\frac{E_1'}{c} = F'^{10} = \Lambda^1{}_\alpha \Lambda^0{}_\beta F^{\alpha\beta}$$

We can ignore terms with $\alpha = \beta$ since $F^{\alpha\alpha} = 0$

For any specific value of δ at most two non-zero terms in $\Lambda^\delta{}_\gamma$.

$$\frac{E_1'}{c} = \Lambda^1{}_0 \Lambda^0{}_1 F^{01} + \Lambda^1{}_1 \Lambda^0{}_0 F^{10}$$

$$= (-\gamma\beta)(-\gamma\beta)\left(-\frac{E_1}{c}\right) + \gamma \cdot \gamma \frac{E_1}{c}$$

$$= \gamma^2(1-\beta^2)\frac{E_1}{c}$$

$$= \frac{E_1}{c}$$

$$\frac{E_2'}{c} = F'^{20} = \Lambda^2{}_\alpha \Lambda^0{}_\beta F^{\alpha\beta}$$

$$= \Lambda^2{}_2 \left(\Lambda^0{}_0 F^{20} + \Lambda^0{}_1 F^{21} \right)$$

$$= \left(\gamma \frac{E_2}{c} + (-\gamma\beta) B_3 \right)$$

$$= \gamma \left(\frac{E_2}{c} - \beta B_3 \right)$$

$$B'_1 = F'^{32} = \Lambda^3{}_\alpha \Lambda^2{}_\beta F^{\alpha\beta}$$

$$= \Lambda^3{}_3 \Lambda^2{}_2 F^{32}$$

$$= B_1$$

$$B'_2 = F'^{13} = \Lambda^1{}_\alpha \Lambda^3{}_\beta F^{\alpha\beta}$$

$$= \Lambda^3{}_3 \left(\Lambda^1{}_0 F^{03} + \Lambda^1{}_1 F^{13} \right)$$

$$= \left((-\gamma\beta)\left(-\frac{E_3}{c}\right) + \gamma B_2 \right)$$

$$= \gamma\left(B_2 + \beta \frac{E_3}{c} \right)$$

Exercise

Show similarly that:

$$\frac{E_3'}{c} = \gamma\left(\frac{E_3}{c} + \beta B_2\right) \quad \text{and} \quad B_3' = \gamma\left(B_3 - \beta \frac{E_2}{c}\right)$$

Apply these general transformations to the case of our point charge q.

Note: We need the inverse transformations S in terms of S'

$\therefore$ Set $\beta \to -\beta$ in the equations we have just worked out.

Since $\underline{B}' = 0$ we can write:

$$\frac{E_1}{c} = \frac{E_1'}{c} = \frac{q}{4\pi\varepsilon_0 c} \frac{\gamma[x^1 - \beta x^0]}{(R')^3} = \frac{q}{4\pi\varepsilon_0 c} \frac{\gamma[x^1 - \beta x^0]}{\left[(\gamma[x^1 - \beta x^0])^2 + (x^2)^2 + (x^3)^2\right]^{3/2}}$$

$$\frac{E_2}{c} = \gamma \frac{E_2'}{c} = \frac{q}{4\pi\varepsilon_0 c} \gamma \frac{x^2}{(R')^3}$$

$$\frac{E_3}{c} = \gamma \frac{E_3'}{c} = \frac{q}{4\pi\varepsilon_0 c} \gamma \frac{x^3}{(R')}$$

consistent with (15.3) & (15.4)

Magnetic fields in frame S

$$\left.\begin{aligned} B_1 &= B_1' = 0 \\ B_2 &= \gamma\left(B_2' - \beta\frac{E_3'}{c}\right) = -\gamma\beta\frac{E_3'}{c} = -\beta\frac{E_3}{c} \\ B_3 &= \gamma\left(B_3' + \beta\frac{E_2'}{c}\right) = \gamma\beta\frac{E_2'}{c} = \beta\frac{E_2}{c} \end{aligned}\right\} \text{as in } (15.8)$$

again $$\underline{B} = \frac{1}{c}\left(\underline{\beta} \times \underline{E}\right) \qquad (\text{as in } 15.9)$$